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Abstract

We propose a commodity pricing model that extends the Gibson–Schwartz two-factor model to incorporate the effect of linear relations among commodity spot prices and to show the conditions under which the prices are cointegrated. We derive futures and call option prices for the proposed model and indicate that, unlike in Duan and Pliska (2004), the linear relations among commodity prices should affect commodity derivative prices, even when the volatilities of commodity returns are constant. Using crude oil and heating oil market data, we estimate the model and apply the results to the hedging of long-term futures using short-term ones. We also discuss the relationships among futures prices implied by the proposed model, as well as a possible generalization with multilinear relations and seasonality.

Keywords: cointegration, commodity prices, convenience yield, energy.

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1 Introduction

Economies are full of equilibrium relations and comovements. These include, for example, purchasing power parity, covered or uncovered interest rate parity, spot–forward relations, money demand equations, consumption spending, and relations among commodity prices. Although these relations are widely known, they do not seem to be adequately utilized in finance, especially in the area of derivative valuations.

These relations have been modeled using cointegration techniques, which were first implicitly used by Davidson, Hendry, Srba, and Yeo (1978) and later established by Engle and Granger (1987). Cointegration refers to a property that holds between two or more nonstationary time series variables. That is, if certain linear combinations of several nonstationary variables are stationary, these variables are said to be cointegrated. Cointegration is interpreted as a long-term relationship or an equilibrium between variables. This is because cointegrated variables are tied to each other to keep certain linear combinations stationary, and hence they tend to move together. Thus, it is natural to consider whether and how such comovements among cointegrated variables affect the prices of their derivatives.

Although academic papers that analyze cointegration relationships among economic variables are plentiful, research on derivative pricing using cointegration is limited. To the best of our knowledge, Duan and Pliska (2004) were the first to use cointegration in examining derivative pricing. They focused on stocks and priced their options under an assumption termed the local risk-neutral valuation relationship, which by definition implies that the drift terms of stock returns are equal to the risk-free rate under the risk-neutral probability. In this setting, they concluded that cointegration affects option prices only when volatilities are stochastic.

Commodity prices, however, behave differently from stock prices. They are strongly affected by production and inventory conditions, and tend to deviate temporarily from the prices that would exist without those effects. These characteristics were recognized from the theory of storage by Kaldor (1939) and Working (1949). To incorporate such temporary deviations, the concept of convenience yield was introduced, which is a crucial element in commodity pricing models. For example, Gibson and Schwartz (1990, 1993) proposed a two-factor model with commodity spot prices and mean reverting convenience yields, and priced commodity futures and options. Schwartz (1997) investigated three different (one-, two-, and three-factor) models including the Gibson–Schwartz two-factor model, using data for crude oil, gold, and copper prices, and analyzed their long-term hedging strategies. Schwartz and Smith (2000) modeled commodity dynamics in a different setting using long- and short-term factors and found that their model was equivalent to the Gibson–Schwartz model. Many other models have generalized the above, including those of Miltersen and Schwartz (1998), Nielsen and Schwartz (2004), and Casassus and Collin-Dufresne (2005).

When a convenience yield exists, the drift in commodity prices may deviate from the risk-free rate even under risk-neutral probability. Thus, in standard commodity pricing models, Duan and Pliska's (2004) risk-neutral valuation framework does not hold, and their results cannot be directly applied to commodity derivative pricing. This is why we need to extend Duan and Pliska's (2004) framework and investigate commodity pricing using cointegration or, more generally, linear relations among the logarithms of commodity prices.

For this purpose, we generalize the Gibson–Schwartz two-factor model by explicitly incorporating linear relations among commodity prices, which includes cointegration under certain conditions. More specifically, we formulate a commodity pricing model in which the temporary deviation of drift terms from the risk-free rate under a risk-neutral probability is described by convenience yields and linear relations among commodity prices, which correspond to error correction terms under appropriate conditions. In previous studies, such temporary deviations were modeled using only the convenience yield; therefore, this paper also can be regarded as proposing a model that specifies a part of the temporary deviation of commodity prices by their cointegrating relationship.

Intuitively, we can expect that relations among commodity prices should characterize part of the deviation for the following reason. As explained, in standard commodity pricing models, drifts in commodity returns may deviate from the risk-free rate. Such deviations are thought to occur because of frictions (e.g. nonnegative constraints and/or transaction costs) in commodity trading. However, if the deviation occurs because of such frictions, then temporary deviations from the long-term relation among commodity prices may not dissolve immediately, either. Consequently, the relations among commodity prices may affect the deviation in addition to "convenience."

It is important to note that several previous studies on commodities incorporated linear relations among prices, or cointegration, into their pricing models. Dempster, Medova, and Tang (2008) analyzed spread options on two commodity prices, assuming that the spread was stationary. However, they did not explicitly model the spot prices, instead directly modeling the spread. This approach simplified their model, but it does not enable us to value futures and options on each commodity, whose prices are cointegrated.

Cortazar, Milla, and Severino (2008) developed a general multicommodity model in which prices of commodities share a set of common factors, through which movements of different commodity prices are related. Such a relation among commodity prices should then provide useful information for describing the movement of each commodity price more accurately. Using data on WTI oil and Brent oil and data on WTI oil and gasoline, Cortazar et al. (2008) assessed multicommodity models and found them to be superior to traditional individual-commodity models.

Based on a similar idea, Paschke and Prokopczuk (2009) also developed a general and tractable multifactor model in which commodity spot prices are characterized by the weighted sum of latent factors. Using NYMEX data for crude oil, heating oil, and gasoline, and unlike Cortazar et al. (2008), they estimated the model simultaneously with three different commodities, and identified the latent factors that jointly characterized those commodity prices.

Adopting a different approach, Casassus, Liu, and Tang (2009) modeled long-term relationships among commodity prices using an intuition that is essentially the same as cointegration. They estimated their model using market data for spread options and compared their results with existing models.

This paper adopts the same approach as those papers, but focuses on a different aspect of commodity price dependencies. Moreover, this paper derives the condition for cointegration, which is not discussed in the abovementioned papers. In the following, we investigate the effect of linear relations among spot prices on commodity derivative pricing for which Duan and Pliska's (2004) risk-neutral valuation does not hold. More precisely, based on Duan and Pliska's (2004) framework, we formulate the Gibson–Schwartz two-factor model with linear relations among commodity spot prices, or cointegration, under certain conditions. We obtain an analytical formula for commodity futures and options prices, and then investigate empirically the effect of such spot price relationships on derivative prices using crude oil and heating oil data from NYMEX. The results suggest that the linear relation between crude oil and heating oil prices partly explains the deviation of drifts in their returns from the risk-free rate, and hence affects their derivative prices.

The rest of this paper is organized as follows. In Section 2, we model commodity spot prices and convenience yields using linear relations among the logarithms of commodity prices with an error correction term in the drift of spot prices. We also investigate the relationship between our model and the Gibson–Schwartz model, and derive the closed-form pricing formulae of futures and call options. In Section 3, we show the state equation and observation equation for the Kalman filter, and conduct an empirical analysis using crude oil and heating oil data. Section 4 discusses the results and Section 5 concludes.

2 The Model

2.1 Gibson–Schwartz Two-Factor with Cointegration (GSC) Model

We propose a model that extends the Gibson–Schwartz two-factor model (hereafter, the GS model; Gibson and Schwartz, 1990; Schwartz, 1997) to explicitly incorporate linear relations among commodity prices, or cointegration, under certain conditions. We adopt the continuous-time specification of cointegrated systems shown by Duan and Pliska (2004).[1] As usual in commodity pricing models, we start by describing the behavior of spot prices and convenience yields under the risk-neutral probability.

Assume that there are n commodities whose spot prices and convenience yields under the risk-neutral probability are as follows:[2]

$$
\begin{aligned}
d\ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) + b_i z(t)\right) dt + \sigma_{S_i} dW_{S_i}(t) && (1)\\
d\delta_i(t) &= \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t), \quad i = 1, \ldots, n. && (2)
\end{aligned}
$$

Here, r is the risk-free interest rate, which is assumed to be constant. b_i, σ_{S_i}, κ_i, $\hat{\alpha}_i$, and σ_{δ_i} are constant coefficients. $\boldsymbol{W}(t) = [W_{S_1}(t), ..., W_{S_n}(t), W_{\delta_1}(t), ..., W_{\delta_n}(t)]^\top$

[1] Duan and Pliska (2004) considered stock prices where $\delta_i(t) = 0$ (no convenience yield for stocks), and showed that the diffusion limit of discrete stock price processes with cointegration among their log prices $lnS_i(t)$ is given by $dS_i(t) = S_i(t)(r+\lambda_i\sigma_{S_i}+b_i z(t))dt+ S_i(t)\sigma_{S_i} dW_{S_i}^P(t)$ under the natural probability where λ_i is the market price of risk.

[2] In Subsection 5.2, we discuss how we can enhance our model to incorporate seasonality.

is a $2n$-dimensional Brownian motion under the risk-neutral probability with

$$dW_{S_i}(t)dW_{S_j}(t) = \rho_{S_iS_j}dt, dW_{S_i}(t)dW_{\delta_j}(t) = \rho_{S_i\delta_j}dt, dW_{\delta_i}(t)dW_{\delta_j}(t) = \rho_{\delta_i\delta_j}dt \quad i,j = 1,\ldots,n.$$

We assume that the commodity prices are related linearly through

$$z(t) = \mu_z + a_0 t + \sum_{i=1}^{n} a_i \ln S_i(t), \tag{3}$$

where μ_z, a_0, and a_is are constants. If $\ln S_i$ are cointegrated, then by rearranging the equation as $\ln S_1(t) = (-\mu_z - a_0 t - \sum_{i=2}^{n} a_i \ln S_i(t) + z(t))/a_1$, if $a_1 \neq 0$, $z(t)$ can be interpreted as an error correction term, a_i as cointegration vectors, and b_i as adjustment speeds of the error correction term. Using Ito's lemma, the dynamics of $z(t)$ is

$$\begin{aligned} dz(t) &= a_0 dt + \sum_{i=1}^{n} a_i d\ln S_i(t) \\ &= \left(a_0 + \sum_{i=1}^{n} a_i r - \frac{1}{2}\sum_{i=1}^{n} a_i \sigma_{S_i}^2 - \sum_{i=1}^{n} a_i \delta_i(t) + \sum_{i=1}^{n} a_i b_i z(t) \right) dt \\ &\quad + \sum_{i=1}^{n} a_i \sigma_{S_i} dW_{S_i}(t). \end{aligned} \tag{4}$$

Define $b = \sum_{i=1}^{n} b_i a_i$. If $b \neq 0$, the above equation can be written as

$$\begin{aligned} dz(t) &= -b(m - z(t))dt - \sum_{i=1}^{n} a_i \delta_i(t) dt + \sum_{i=1}^{n} a_i \sigma_{S_i} dW_{S_i}(t) \\ m &= \frac{-a_0 - \sum_{i=1}^{n} a_i r + \frac{1}{2}\sum_{i=1}^{n} a_i \sigma_{S_i}^2}{b}. \end{aligned} \tag{5}$$

The set of equations (1), (2), and (3) is an extension of the GS model with a linear relation $z(t)$ among the logarithms of commodity prices that affects the drift terms. $z(t)$ represents the error correction term of the cointegrating relationship among commodity prices.[3] We call this model the Gibson–Schwartz two-factor with cointegration model (hereafter the GSC model).

[3]In Subsection 2.3, we show a sufficient condition for cointegration.

It is worth mentioning that while the GSC model bases its specification on Duan and Pliska (2004), the drift term is different from that of Duan and Pliska (2004), in which the drift is equal to the risk-free rate under the risk-neutral probability. This difference comes from the characteristics of the underlying assets. For stocks, which Duan and Pliska (2004) focused on, it is natural to assume that the drift terms of returns should be equal to the risk-free rate under the risk-neutral probability. On the other hand, for commodities, it is standard to assume that the drift terms may deviate temporarily from the risk-free rate even under the risk-neutral probability by reflecting inventory and production conditions. The GSC model assumes that such deviations are described by the convenience yield and the term $z(t)$.[4]

2.2 Futures and Option Prices for the GSC Model

We derive the futures and European call option prices on commodity i in matrix form.

$$\begin{aligned}
d\ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) + b_i\mu_z + b_i a_0 t + \sum_{j=1}^{n} b_i a_j \ln S_j(t)\right) dt + \sigma_{S_i} dW_{S_i}(t) \\
&\equiv \left(\beta_{S_i0}(t) + \beta_{S_i\delta_i}\delta_i(t) + \sum_{j=1}^{n} \beta_{S_iS_j} \ln S_j(t)\right) dt + \sigma_{S_i} dW_{S_i}(t) \\
d\delta_i(t) &\equiv (\beta_{\delta_i0} + \beta_{\delta_i\delta_i}\delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t),
\end{aligned}$$

where

$$\begin{aligned}
\beta_{S_i0}(t) &= r - \frac{\sigma_{S_i}^2}{2} + b_i\mu_z + b_i a_0 t \\
\beta_{S_iS_j} &= b_i a_j \\
\beta_{S_i\delta_i} &= -1 \\
\beta_{\delta_i0} &= \kappa_i\hat{\alpha}_i \\
\beta_{\delta_i\delta_i} &= -\kappa_i.
\end{aligned}$$

[4] See Section 1 for the intuition behind including the linear relation $z(t)$ in the drift terms.

This equation can be solved as follows.[5]

$$\boldsymbol{X}(T) = e^{(T-t)\boldsymbol{\beta}} \left\{ \boldsymbol{X}(t) + \int_t^T e^{-s\boldsymbol{\beta}} \boldsymbol{\beta}_0(s) ds + \int_t^T e^{-s\boldsymbol{\beta}} d\boldsymbol{W}_0(s) \right\}, \quad (6)$$

where

$$\begin{aligned}
\boldsymbol{X}(t) &= [\ln S_1(t), \cdots, \ln S_n(t), \delta_1(t), \cdots, \delta_n(t)]^\top \\
\boldsymbol{\beta}_0(t) &= [\beta_{S_1 0}(t), \cdots, \beta_{S_n 0}(t), \beta_{\delta_1 0}, \cdots, \beta_{\delta_n 0}]^\top \\
\boldsymbol{\beta} &= \begin{bmatrix} \beta_{S_1 S_1} & \cdots & \beta_{S_1 S_n} & \beta_{S_1 \delta_1} & & 0 \\ \vdots & \ddots & \vdots & & \ddots & \\ \beta_{S_n S_1} & \cdots & \beta_{S_n S_n} & 0 & & \beta_{S_n \delta_n} \\ & & & \beta_{\delta_1 \delta_1} & & 0 \\ & \mathbf{0} & & & \ddots & \\ & & & 0 & & \beta_{\delta_n \delta_n} \end{bmatrix}
\end{aligned}$$

and $\boldsymbol{W}_0(t) = [\sigma_{S_1} W_{S_1}(t), \cdots, \sigma_{S_n} W_{S_n}(t), \sigma_{\delta_1} W_{\delta_1}(t), \cdots, \sigma_{\delta_n} W_{\delta_n}(t)]^\top$ is a scaled Brownian motion vector.

Denote by $E[\cdot]$ the expectation under the risk-neutral probability. The mean and covariances of $\ln S_i(T)$ are

$$\begin{aligned}
\mu_{X_i}(t,T) &= E_t[\ln S_i(T)] \\
&= \left[e^{(T-t)\boldsymbol{\beta}} \left\{ \boldsymbol{X}(t) + \int_t^T e^{-s\boldsymbol{\beta}} \boldsymbol{\beta}_0(s) ds \right\} \right]_i \\
\sigma_{X_i X_j}(t,T) &= E_t[(\ln S_i(T) - \mu_{X_i}(t,T))(\ln S_j(T) - \mu_{X_i}(t,T))] \\
&= \left[\int_t^T (e^{(T-t-s)\boldsymbol{\beta}}) \boldsymbol{\Omega} (e^{(T-t-s)\boldsymbol{\beta}})^\top ds \right]_{ij},
\end{aligned}$$

where $[\cdot]_i$ and $[\cdot]_{ij}$ are the ith vector element and the $[i,j]$th matrix element,

[5] Cf. Karatzas and Shreve (1991), Section 5.6 or Liptser and Shiryaev (2001), p. 151, Thm. 4.10.

respectively, and the covariance matrix is

$$\Omega = \begin{bmatrix} \rho_{S_1S_1}\sigma_{S_1}\sigma_{S_1} & \cdots & \rho_{S_1S_n}\sigma_{S_1}\sigma_{S_n} & \rho_{S_1}\sigma_{S_1}\sigma_{\delta_1} & \cdots & \rho_{S_1\delta_n}\sigma_{S_1}\sigma_{\delta_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{S_nS_1}\sigma_{S_n}\sigma_{S_1} & \cdots & \rho_{S_nS_n}\sigma_{S_n}\sigma_{S_n} & \rho_{S_n\delta_1}\sigma_{S_n}\sigma_{\delta_1} & \cdots & \rho_{S_n\delta_n}\sigma_{S_n}\sigma_{\delta_n} \\ \rho_{\delta_1S_1}\sigma_{\delta_1}\sigma_{S_1} & \cdots & \rho_{\delta_1S_n}\sigma_{\delta_1}\sigma_{S_n} & \rho_{\delta_1\delta_1}\sigma_{\delta_1}\sigma_{\delta_1} & \cdots & \rho_{\delta_1\delta_n}\sigma_{\delta_1}\sigma_{\delta_n} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \rho_{\delta_nS_1}\sigma_{\delta_n}\sigma_{S_1} & \cdots & \rho_{\delta_nS_n}\sigma_{\delta_n}\sigma_{S_n} & \rho_{\delta_n\delta_1}\sigma_{\delta_n}\sigma_{\delta_1} & \cdots & \rho_{\delta_n\delta_n}\sigma_{\delta_n}\sigma_{\delta_n} \end{bmatrix}.$$

Using risk neutrality and properties of a moment generating function, we obtain the futures price of commodity i as follows (cf. Cox, Ingersoll, and Ross, 1981).

Proposition 2.1. *Assuming (1), (2), and (3), the futures price of commodity i with maturity T at t is given by*

$$\begin{aligned} G_i(t,T) &= E_t[S_i(T)] \\ &= \exp\left\{\mu_{X_i}(t,T) + \frac{\sigma^2_{X_i}(t,T)}{2}\right\}, \end{aligned}$$

where $\sigma^2_{X_i}(t,T) = \sigma_{X_iX_i}(t,T)$.

Note that there is $\ln S_i(t)$ in $\mu_{X_i}(t,T)$ implicitly, so $S_i(t)$ do not appear in the formula.

In the following proposition, we derive the call option pricing formula. This is not addressed by other papers that incorporate multicommodity prices, such as Cortazar et al. (2008), Paschke and Prokopczuk (2009), and Casassus et al. (2009).

Proposition 2.2. *Assuming (1), (2), and (3), the European call option price of commodity i with maturity T at t is given by*

$$\begin{aligned} C_i(t,T) &= e^{-r(T-t)}E_t[(S_i(T)-K)^+] \\ &= e^{-r(T-t)+\mu_{X_i}(t,T)+\frac{\sigma^2_{X_i}(t,T)}{2}}\Phi(d_{i1}(t,T)) - Ke^{-r(T-t)}\Phi(d_{i2(t,T)}) \\ d_{i1}(t,T) &= \frac{-\ln K + \mu_{X_i}(t,T) + \sigma^2_{X_i}(t,T)}{\sigma_{X_i}(t,T)} \\ d_{i2}(t,T) &= d_{i1}(t,T) - \sigma_{X_i}(t,T). \end{aligned}$$

Proof. See the Appendix for the derivation. □

In the Appendix, we elaborate the formulae of the derivatives without using integrals or matrices. These formulae are more tractable than the matrix formulae in this subsection for calculating the hedge weights and applying them to risk management.

2.3 A Sufficient Condition for Cointegration

We now show a sufficient condition for the GSC model to be cointegrated.

Proposition 2.3. *Let us assume* $e^{b\triangle t} < 1(\iff b < 0), e^{-\kappa_i \triangle t} < 1(\iff \kappa_i > 0)$. *Then, the GSC model is cointegrated.*

Proof. In the Appendix. □

This condition is very important for the estimated parameters in the model to be valid. If the condition of Proposition 2.3 does not hold, the estimation of the model may lead to a spurious regression. The estimated coefficients in this case are not consistent and the sample residual of $z(t)$ will be nonstationary.[6] This condition is similar to that of Duan and Pliska (2004). However, because our setting is different from their model, we cannot simply apply their results. In particular, convenience yields are unobservable in our model.[7] Therefore, we need a different condition for cointegration for our model. To our knowledge, among the papers that deal with relations among commodity prices, including Cortazar et al. (2008), Paschke and Prokopczuk (2009), and Casassus et al. (2009), ours is the first to show a sufficient condition for cointegration.

[6] See Hamilton (1994) for the properties of spurious regressions.

[7] For unobservability, see Section 5.1.

3 Empirical Analysis

3.1 The Dynamics of Commodity Spot Prices, Convenience Yields, and Error Terms under Natural Probability

As neither commodity spot prices nor convenience yields are observable, we have to estimate their parameters using the Kalman filter[8] with their futures prices. We have already modeled commodity spot prices and convenience yields under the risk-neutral probability, and thus, we can calculate the Kalman filter with only these SDEs. However, because it would be useful to check whether the model performs well under the natural probability, the SDEs of commodity spot prices and convenience yields under the natural probability are needed to estimate the model. For this purpose, we assume the market price of risk that transforms the risk-neutral probability into the natural probability.

Let us assume that Brownian motions under the risk-neutral probability $W(t)$ and Brownian motions under the natural probability $W^P(t)$ satisfy

$$
\begin{aligned}
&\boldsymbol{W}(t) = \boldsymbol{W}^P(t) + \int_0^t \boldsymbol{\theta}_0 ds \\
&\boldsymbol{W}(t) = [W_{S_1}(t), \cdots, W_{S_n}(t), W_{\delta_1}(t), \cdots, W_{\delta_n}(t)]^\top \\
&\boldsymbol{W}^P(t) = [W^P_{S_1}(t), \cdots, W^P_{S_n}(t), W^P_{\delta_1}(t), \cdots, W^P_{\delta_n}(t)]^\top \\
&\boldsymbol{\theta}_0 = [\theta_{S_1 0}, \cdots, \theta_{S_n 0}, \theta_{\delta_1 0}, \cdots, \theta_{\delta_n 0}]^\top,
\end{aligned}
$$

where $\boldsymbol{\theta}$ is the market price of risk, which is assumed to be constant. The consequence of this assumption can be seen in the following SDEs under the natural probability.[9]

$$
\begin{aligned}
d\ln S_i(t) &= \left(\gamma_{S_i 0}(t) + \sum_{j=1}^n \gamma_{S_i S_j} \ln S_j(t) + \gamma_{S_i \delta_i}\delta_i(t)\right) dt \\
&\quad + \sigma_{S_i} dW^P_{S_i}(t) \qquad (7) \\
d\delta_i(t) &= \left(\gamma_{\delta_i 0} + \gamma_{\delta_i \delta_i}\delta_i(t)\right) dt + \sigma_{\delta_i} dW^P_{\delta_i}(t), \qquad (8)
\end{aligned}
$$

[8]For the Kalman filter, see Hamilton (1994).

[9]The solution for SDEs (7) and (8), the state equation, the observation equation, the Kalman filters, forecasts, and the maximum likelihood for this model are available on request.

where

$$\begin{aligned}
\gamma_{S_i 0}(t) &= r - \frac{\sigma_{S_i}^2}{2} + b_i\mu_z + b_i a_0 t + \sigma_{S_i}\theta_{S_i 0} \\
\gamma_{S_i S_j} &= b_i a_j \\
\gamma_{S_i \delta_i} &= -1 \\
\gamma_{\delta_i 0} &= \kappa_i \hat{\alpha}_i + \sigma_{\delta_i}\theta_{\delta_i 0} \\
\gamma_{\delta_i \delta_i} &= -\kappa_i
\end{aligned}$$

To implement the empirical analysis, for ease of calculation, we classify the model into five cases, estimate each of them separately, and compare them. The cases are: (i) $b \neq 0$ and $a_1 = 1$, (ii) $b \neq 0$ and $a_2 = 1$, (iii) $b = 0, a_1 = 1, b_1 = -a_2 b_2$, (iv) $b = 0, a_1 = 0, a_2 = 0$, and (v) $b = 0, a_1 = 0, b_2 = 0$.[10] This enables us to calculate the log-likelihood using the scalar forms and to avoid the time-consuming calculation of the general case using the matrix formula. In this subsection, we show the result when $b \neq 0$ and $a_2 = 1$.[11] For the other results, see the Appendix. The GS model is enhanced to have two commodity price processes and two convenience yield processes, with correlations between each process.

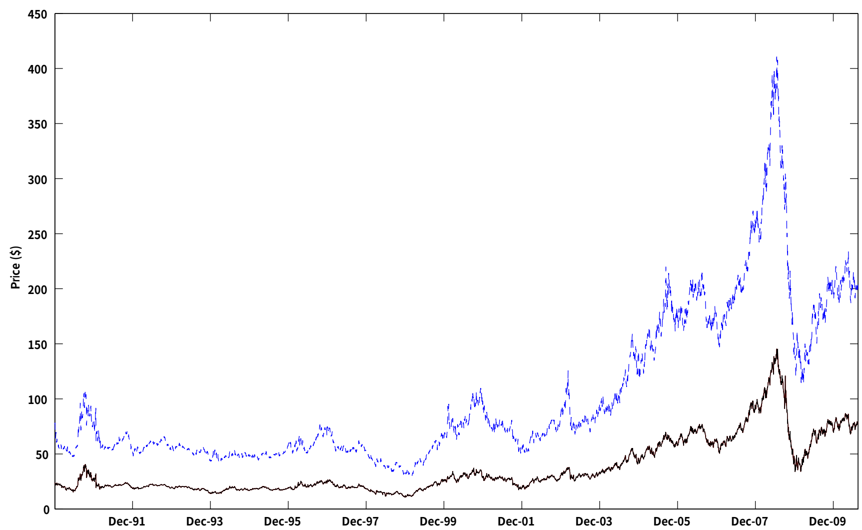
3.2 Data

We use WTI and heating oil daily closing prices traded on the NYMEX from January 2, 1990, to July 30, 2010. Five futures contracts, labeled Maturity 1, Maturity 3, Maturity 5, Maturity 7, Maturity 9, are used in the estimation. Maturity 1 stands for the contract closest to maturity, Maturity 3 stands for the third closest maturity, and so on. The time to maturity corresponding to these prices is also used. We set the risk-free rate equal to 4%.

[10] These cases are collectively exhaustive. This can be shown as follows. There are only two cases: $b \neq 0$ and $b = 0$. For $b \neq 0$, there are only two cases, $a_1 \neq 0$ or $a_2 \neq 0$, because otherwise we have $b = 0$. Furthermore, $a_1 \neq 0$ or $a_2 \neq 0$ can be rescaled to 1, case (i) or (ii), respectively. For $b = 0$, we have $a_1 \neq 0$, which can be rescaled to $a_1 = 1$, which is case (iii), or $a_1 = 0$. If $a_1 = 0$, then there are two possibilities, which are $a_2 = 0$ or $b_2 = 0$, case (iv) or (v), respectively; otherwise $b = a_2 b_2 \neq 0$. Furthermore, note that cases (i) and (ii) may satisfy the cointegration condition ($b < 0$), but the other cases do not because $b = 0$.

[11] In our empirical analysis, this case is found to have the smallest AIC among the cases that satisfy the cointegration condition in Proposition 2.3.

Figure 1: WTI and heating oil daily closing prices from January 2, 1990, to July 30, 2010. The black solid line and the blue dashed line denote the prices of crude oil and heating oil, respectively.



The basic statistics for these data are described in Table 1. As the maturity dates are fixed, the time to maturity changes over time. Comparing WTI crude oil with heating oil, we can see that the standard deviation of heating oil is higher because the average price of heating oil is higher than that of crude oil. The mean maturity and its standard deviation are quite close to each other. Furthermore, note that the correlation between the futures prices of WTI and heating oil is 0.995.

Table 1: Data statistics.

Futures contract	Mean price (Standard deviation)	Mean log return (Standard deviation)	Mean maturity (Standard deviation)
WTI crude oil			
Maturity 1	36.69 (25.02)	0.0240 % (2.5417 %)	0.10 (0.04)
Maturity 3	36.79 (25.43)	0.0250 % (2.0547 %)	0.35 (0.04)
Maturity 5	36.71 (25.72)	0.0259 % (1.8571 %)	0.59 (0.04)
Maturity 7	36.60 (25.93)	0.0267 % (1.7382 %)	0.83 (0.04)
Maturity 9	36.48 (26.08)	0.0272 % (1.6528 %)	1.08 (0.04)
Heating oil			
Maturity 1	101.67 (70.14)	0.0188 % (2.4679 %)	0.09 (0.04)
Maturity 3	102.23 (71.50)	0.0240 % (2.0456 %)	0.34 (0.04)
Maturity 5	102.40 (72.58)	0.0260 % (1.8614 %)	0.58 (0.04)
Maturity 7	102.36 (73.35)	0.0264 % (1.7512 %)	0.82 (0.04)
Maturity 9	102.17 (73.68)	0.0260 % (1.6702 %)	1.07 (0.04)

3.3 Estimation Results

We estimate the model using the Kalman filter. In Table 2, we report the estimated parameters with standard errors. Note that the AIC for the GSC model is lower than that for the GS model, which implies that the GSC model fits the data better. As we can see, the estimated linear relation vectors are $[a_1, a_2] = [-1.19, 1.00]$ and the adjustment speeds are $[b_1, b_2] = [-0.05, -0.36]$, respectively. A comparison of these values with the standard errors suggests that the linear relation among commodity prices empirically affects the derivative prices.

As the values of b_1, b_2 measure how much the linear relation affects the spot prices, it also suggests that the heating oil price is much more affected by the linear relation, or the error correction term, than the crude oil price is. Note that a_0 is -0.000072 and its standard error is 0.000004, which implies that the relation term $z(t)$ includes time drift, but μ_z is small compared with the standard error. Furthermore, b is -0.29. With both κ_i positive, the cointegration conditions are satisfied. Therefore, we can compare the coefficients with their standard deviations to check the significance of the coefficients.

Except $\hat{\alpha}_1$ in the GS model, $\hat{\alpha}_i$ are significant. However, they are different between the two models. This is a result of the relation term $z(t)$. As mentioned above, the GSC model assumes that the deviation of the drift terms from the risk-free rate is described by the convenience yield and the term $z(t)$. Thus, it is only a matter of which factor explains most of the deviation, and $\hat{\alpha}_i$ depends on these factors. Both κ_i exceed 1, which is the same as in the GS model.

Let us turn now to the volatility parameters. In the GSC model, crude oil and heating oil spot prices have a positive correlation ($\rho_{S_1 S_2} = 0.75$). The corresponding spot prices and convenience yields have relatively high positive correlations $\rho_{S_1\delta_1} = 0.77$ and $\rho_{S_2\delta_2} = 0.62$, respectively, which is consistent with the GS model. Moreover, crude oil spot prices and heating oil convenience yields have no correlation ($\rho_{S_1\delta_2} = 0.00$). However, we see that the correlation for heating oil spot prices and crude oil convenience yields $\rho_{S_2\delta_1}$ is relatively high (0.63). This is the same as in the GS model. It is intuitive that spot prices and convenience yields among different commodities should not be strongly correlated; however, heating oil prices are affected by crude oil convenience yields. Volatilities of spot prices σ_{S_1}, σ_{S_2} and convenience yields σ_{δ_1}, σ_{δ_2} seem to be similar in the two models.

Table 3 shows the root mean square error (RMSE) and mean error (ME) of the model. Although both models have small values, which indicates that the models are well fitted, the RMSE and ME both favor the GSC model with few exceptions.

Table 2: Estimated parameters, with standard errors in parentheses. Data are WTI and heating oil daily closing prices traded on the NYMEX from January 2, 1990, to July 30, 2010.

	GS	GSC
Volatility parameters		
σ_{S_1}	0.414476 (0.003512)	0.381896 (0.002241)
σ_{S_2}	0.377914 (0.002247)	0.406307 (0.002546)
σ_{δ_1}	0.320532 (0.002552)	0.287109 (0.001652)
σ_{δ_2}	0.507958 (0.005171)	0.699693 (0.007743)
$\rho_{S_1 S_2}$	0.698858 (0.005467)	0.748660 (0.004673)
$\rho_{S_1 \delta_1}$	0.793308 (0.004015)	0.767305 (0.004128)
$\rho_{S_1 \delta_2}$	0.000058 (0.012618)	0.000072 (0.012604)
$\rho_{S_2 \delta_1}$	0.505952 (0.005636)	0.628424 (0.005645)
$\rho_{S_2 \delta_2}$	0.600362 (0.009234)	0.620154 (0.008083)
$\rho_{\delta_1 \delta_2}$	0.108853 (0.011792)	0.165843 (0.014062)
Convenience yield parameters		
κ_1	1.070822 (0.005328)	1.140883 (0.006597)
κ_2	1.294663 (0.014874)	1.085038 (0.015096)
$\hat{\alpha}_1$	0.001375 (0.001417)	0.006611 (0.003161)
$\hat{\alpha}_2$	0.038074 (0.002409)	-0.037714 (0.020791)
Linear relation parameters		
μ_z		1.144262 (0.046325)
a_0		-0.000072 (0.000004)
a_1		-1.187431 (0.006754)
a_2		1.000000 (n.a.)
b_1		-0.052615 (0.001626)
b_2		-0.356252 (0.005272)
Market price of risk parameters		
$\theta_{S_1 0}$	0.083425 (0.267456)	0.478595 (0.235855)
$\theta_{S_2 0}$	-0.357933 (0.212986)	0.817002 (0.231929)
$\theta_{\delta_1 0}$	0.074827 (0.247218)	-0.002131 (0.241285)
$\theta_{\delta_2 0}$	-0.281003 (0.282557)	-0.351462 (0.269419)
$R(1,1)$	0.000509 (0.000005)	0.000520 (0.000006)
$R(2,2)$	0.000000 (0.000000)	0.000002 (0.000000)
$R(3,3)$	0.000009 (0.000000)	0.000008 (0.000000)
$R(4,4)$	0.000000 (0.000000)	0.000000 (0.000000)
$R(5,5)$	0.000023 (0.000001)	0.000020 (0.000001)
$R(6,6)$	0.000002 (0.000001)	0.000003 (0.000001)
$R(7,7)$	0.001043 (0.000029)	0.001019 (0.000030)
$R(8,8)$	0.000742 (0.000024)	0.000700 (0.000022)
$R(9,9)$	0.000002 (0.000000)	0.000007 (0.000000)
$R(10,10)$	0.001138 (0.000029)	0.000999 (0.000027)
Log-likelihood	153030.832494	154335.136395
AIC	-306005.664988	-308604.272790
Sample size	51590	51590

Table 3: Root mean square error (RMSE) and Mean error (ME) for each futures contract.

Contracts	RMSE		ME	
Models	GS	GSC	GS	GSC
Crude oil				
Maturity 1	0.032872	0.032943	-0.002727	-0.002741
Maturity 3	0.020227	0.020206	0.000462	-0.000003
Maturity 5	0.018617	0.018556	0.000565	0.000026
Maturity 7	0.017313	0.017296	0.000434	-0.000030
Maturity 9	0.017211	0.017152	0.000625	0.000170
Heating oil				
Maturity 1	0.024144	0.024064	0.000889	0.000023
Maturity 3	0.037593	0.037136	0.000502	-0.001603
Maturity 5	0.032656	0.031699	0.000677	-0.000226
Maturity 7	0.018059	0.017767	0.000717	-0.000012
Maturity 9	0.038907	0.036419	0.000424	-0.003012

4 Hedging Futures

In this section, we implement the GSC and GS models for hedging long-term futures contracts,[12] which we call the target futures using short-term futures. We empirically analyze the logarithms of the crude oil and heating oil prices.

As the GS model has two stochastic variables, we need two futures that have different maturities to hedge, and the weights can be calculated by

[12] Recall that we are assuming the risk-free rate is constant, which implies that futures and forwards are equally valued.

solving the following system of equations.[13]

$$
\begin{aligned}
\boldsymbol{\Phi} \boldsymbol{w} &= \boldsymbol{\varphi} && (9)\\
\boldsymbol{\Phi} &= \begin{bmatrix} \frac{\partial G_i(t,T_1)}{\partial S_i} & \frac{\partial G_i(t,T_2)}{\partial S_i} \\ \frac{\partial G_i(t,T_1)}{\partial \delta_i} & \frac{\partial G_i(t,T_2)}{\partial \delta_i} \end{bmatrix} \\
\boldsymbol{w} &= [w_1, w_2]^\top && (10)\\
\boldsymbol{\varphi} &= \left[\frac{\partial G_i(t,T)}{\partial S_i}, \frac{\partial G_i(t,T)}{\partial \delta_i} \right]^\top, && (11)
\end{aligned}
$$

where w_i are weights for futures with maturity T_i and T is the maturity of the target futures.

On the other hand, for the GSC model, which has S_i, δ_i, δ_j, and z as stochastic variables,[14] we need four futures to hedge when there are two commodities to consider. Now, the system of equations for (9) is

$$
\begin{aligned}
\boldsymbol{\Phi} &= \begin{bmatrix} \frac{\partial G_i(t,T_1)}{\partial S_i} & \frac{\partial G_i(t,T_2)}{\partial S_i} & \frac{\partial G_i(t,T_3)}{\partial S_i} & \frac{\partial G_i(t,T_4)}{\partial S_i} \\ \frac{\partial G_i(t,T_1)}{\partial \delta_i} & \frac{\partial G_i(t,T_2)}{\partial \delta_i} & \frac{\partial G_i(t,T_3)}{\partial \delta_i} & \frac{\partial G_i(t,T_4)}{\partial \delta_i} \\ \frac{\partial G_i(t,T_1)}{\partial \delta_j} & \frac{\partial G_i(t,T_2)}{\partial \delta_j} & \frac{\partial G_i(t,T_3)}{\partial \delta_j} & \frac{\partial G_i(t,T_4)}{\partial \delta_j} \\ \frac{\partial G_i(t,T_1)}{\partial z} & \frac{\partial G_i(t,T_2)}{\partial z} & \frac{\partial G_i(t,T_3)}{\partial z} & \frac{\partial G_i(t,T_4)}{\partial z} \end{bmatrix} \\
\boldsymbol{w} &= [w_1, w_2, w_3, w_4]^\top \\
\boldsymbol{\varphi} &= \left[\frac{\partial G_i(t,T)}{\partial S_i}, \frac{\partial G_i(t,T)}{\partial \delta_i}, \frac{\partial G_i(t,T)}{\partial \delta_j}, \frac{\partial G_i(t,T)}{\partial z} \right]^\top.
\end{aligned}
$$

We emphasize that we use the futures formula in Proposition A.1 in the Appendix to derive the hedging weights.

To calculate the hedging portfolio, we need the values of state variables $S_i(t)$, $\delta_i(t), \delta_j(t)$, and $z(t)$. There are two methods for calculating the values of the state variables. One method is to use a Kalman filter, which we call the Kalman filter method. The other method is to calculate the values of the state variables by solving the observation equation, which only requires

[13] See Brennan and Crew (1997) and Schwartz (1997) for hedging long-term forwards using short-term futures.

[14] For calculating the hedge weights, we can use S_i, S_j, δ_i, and δ_j instead of S_i, δ_i, δ_j, and z. If we are considering more than three commodities, for example four commodities, then using state variable z is more convenient for calculating hedge weights as we only need $n+2$ futures to hedge, whereas we need $2n$ when using S_i, δ_i.

futures prices and the estimated parameters. We call this method the simultaneous equation method. We implement both methods. The hedging error ratio is calculated by dividing the hedging error value by the target futures price of each hedging start period. The hedging error value is the difference between the target futures price and the value of the hedge portfolio.

We hedge the futures that mature in 1 year and in 10 years with the futures that mature in 1, 3, 5, and 7 months. As long-term futures, e.g. 10-year futures, are not traded in the market, we cannot calculate their hedging error precisely. Hence, to evaluate the hedging error, we also hedge 1-year futures. We calculate the hedging error for 10-year futures by using their theoretical price. The total hedging period is from January 2, 1990, to July 30, 2010. We roll the futures 3 business days before they mature, and each hedging period is roughly 1 month. The hedging weight and the hedging error are calculated daily.

The performance of the hedging simulation for the 1-year futures is indicated in Table 4 and Figure 2. For both commodities, the results indicate that the hedging error ratios are relatively small. This is true for both the GSC and the GS models. Comparing the two models, we see that the GSC model has a relatively good performance using the simultaneous equation method, except for the case of crude oil.

Figure 3 shows the weights of the futures in the hedging portfolio whose state variables are calculated by Kalman filters. For the GS model, the hedge weights for 3-month futures are positive and those for 1-month futures are negative. For the GSC model, the hedge weights for 7-month and 3-month futures are positive and the others are negative.

In Table 5 and Figure 4, we show the performance of the hedging simulation for 10-year futures. Obviously, the hedging error ratio is poorer than that for the 1-year futures. However, note that this hedging error ratio is calculated by using the theoretical price and, therefore, we cannot estimate the hedge errors exactly. Note also that for both commodities, the GS model performs significantly better than the GSC model. For the GSC model, this is because the absolute hedge weight is very large, as indicated in Figure 5. Recall that the hedge weight is calculated as $\boldsymbol{w} = \boldsymbol{\Phi}^{-1}\boldsymbol{\varphi}$. Some of the values in $\boldsymbol{\Phi}$, especially the partial derivatives of the other convenience yield δ_j and z, are too small and hence $\boldsymbol{\Phi}^{-1}$ and the hedge weight are very large. If we erase the partial derivatives for δ_j and z and calculate the hedge weight, the performance of the hedging simulation for 10-year futures improves, but the performance for 1-year futures will not be as good as described above.

Table 4: Performance of hedging 1-year futures. “Kalman filter” indicates that the state variables are calculated using Kalman filters. “Simultaneous” indicates that the state variables are calculated by solving the observation equation.

Contracts	Mean of hedging error ratio	
Method	GS	GSC
Crude oil		
Kalman filter	0.042603	-0.019405
Simultaneous	0.023230	-0.027480
Heating oil		
Kalman filter	0.010182	0.004493
Simultaneous	-0.026031	-0.012187

Figure 2: Performance of hedging 1-year futures. The two graphs on the left show the results of the GS model, and the two graphs on the right show the results of the GSC model. The blue solid line and the red dashed line indicate hedge performance of WTI crude oil and heating oil, respectively.

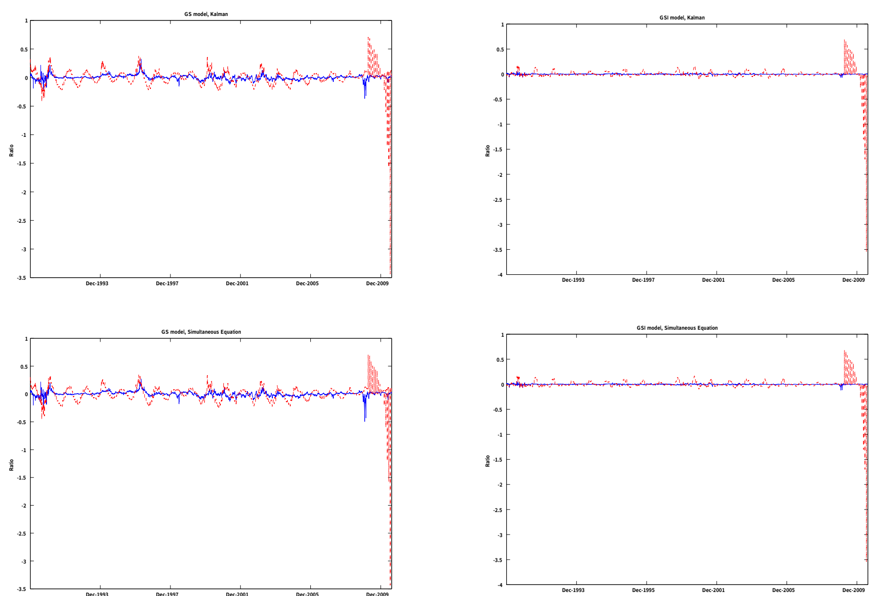
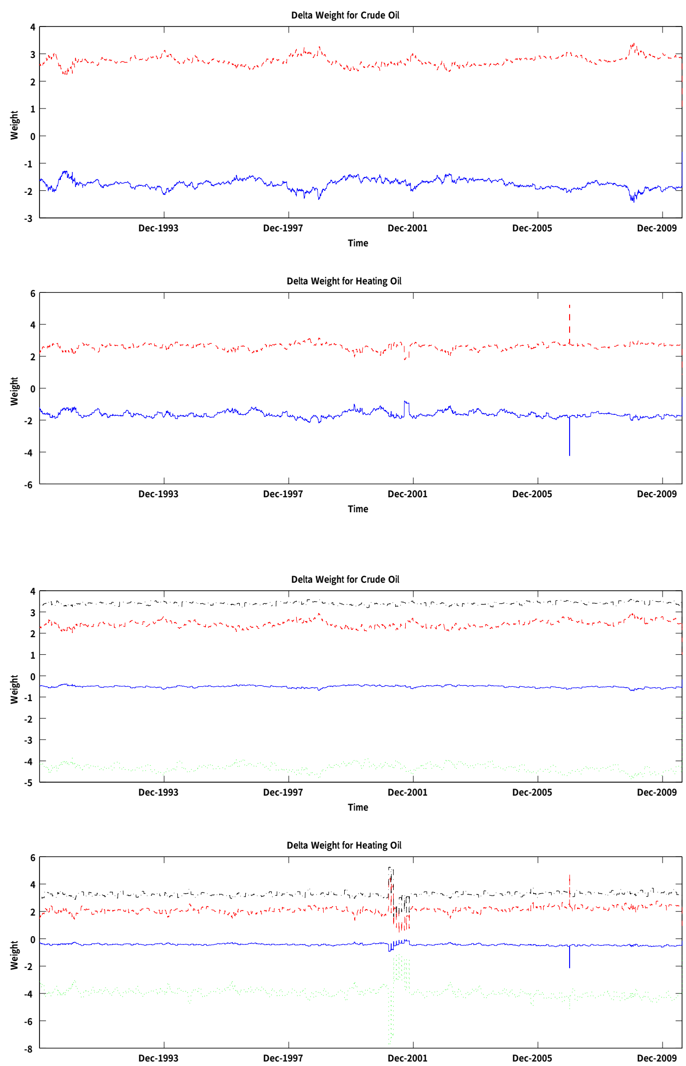


Figure 3: Weights of futures for hedging 1-year futures, for which the state variables are calculated by Kalman filters. The upper two figures show the results for the GS model. The blue solid line and the red dashed line indicate the hedging weights of the 1-month futures and 3-month futures, respectively. The lower two figures show the results for the GSC model. The blue solid line, the red dashed line, the green dotted line, and the black chained line indicate the hedging weights of the 1-month futures, 3-month futures, 5-month futures, and 7-month futures, respectively.

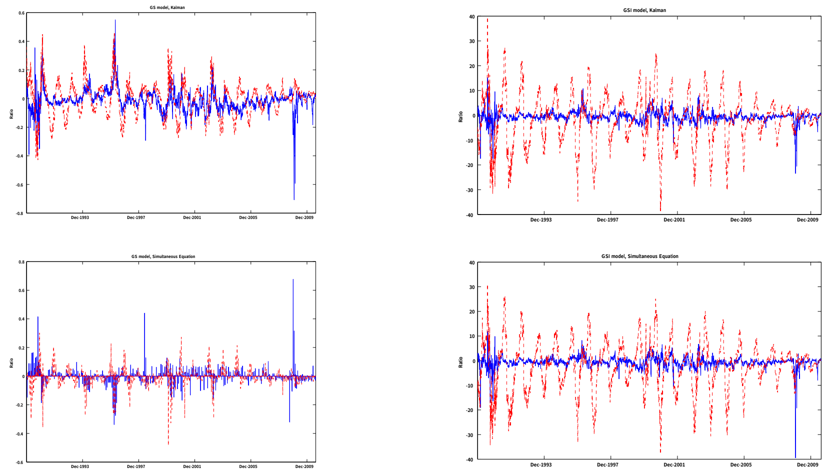


This implies that the Gibson–Schwartz model is good enough for hedging short-term and long-term commodity futures.

Table 5: Performance of hedging 10-year futures. "Kalman filter" indicates that the state variables are calculated using Kalman filters. "Simultaneous" indicates that the state variables are calculated by solving the observation equation.

Contracts	Mean of hedging error ratio	
Method	GS	GSC
Crude oil		
Kalman filter	-0.333988	-12.315692
Simultaneous	0.009467	-13.809328
Heating oil		
Kalman filter	-0.043942	-35.425612
Simultaneous	-0.023776	-42.889075

Figure 4: Performance of hedging 10-year futures. The two graphs on the left present the results of the GS model and the two graphs on the right present the results of the GSC model. The blue solid line and the red dashed line indicate the hedge performance of WTI crude oil and heating oil, respectively.



5 Discussion

5.1 Relations among Futures Prices

It should be noted that, in our setting, the linear relations among commodity spot prices do not automatically apply to the linear relations among their futures prices. Let us look at the dynamics of the logarithms of futures prices. As in Appendix 2, we know that the logarithms of futures prices $\ln G_i(t,T)$ can be represented by

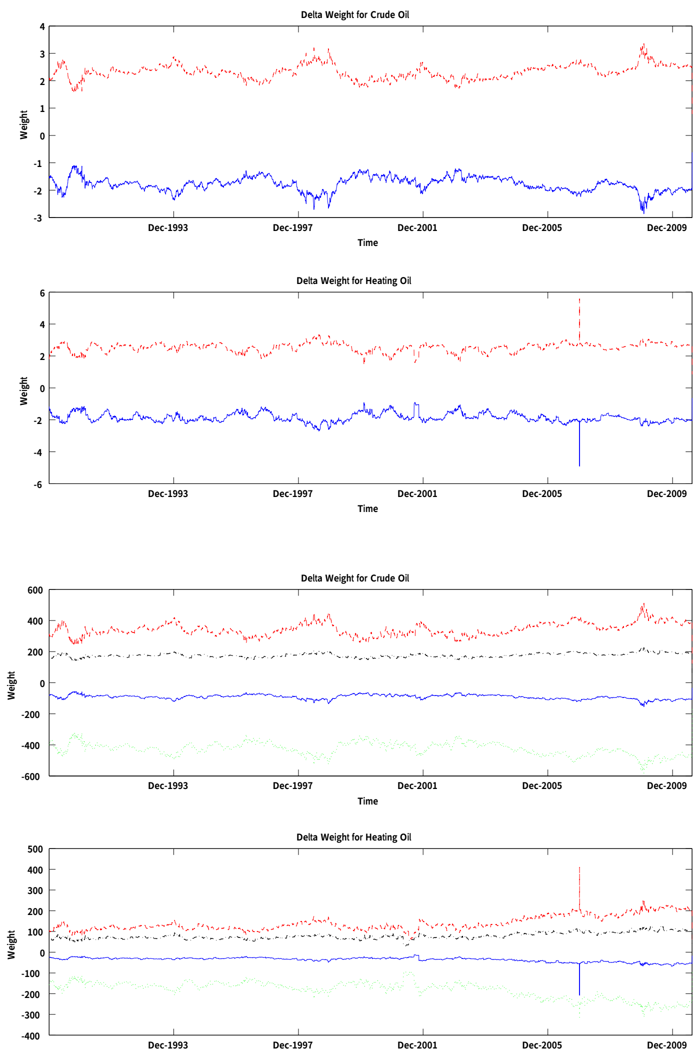
$$\ln G_i(t,T) = \sum_{j=1}^{n} c_{S_j}(t,T) \ln S_j(t) + \sum_{j=1}^{n} c_{\delta_j}(t,T)\delta_j(t) + X'(t,T),$$

where

$$\frac{\partial G_i(t,T)}{\partial \ln S_j(t)} = c_{S_j}(t,T)G_i(t,T), \qquad \frac{\partial G_i(t,T)}{\partial \delta_j(t)} = c_{\delta_j}(t,T)G_i(t,T)$$

and $X'(t,T)$ represents the residual.

Figure 5: Weights of futures for hedging 10-year futures, for which the state variables are calculated using Kalman filters. The upper two figures show the results for the GS model. The blue solid line and the red dashed line indicate the hedging weights of 1-month futures and 5-month futures, respectively. The lower two figures show the result for the GSC model. The blue solid line, the red dashed line, the green dotted line, and the black chained line indicate the hedging weights of 1-month futures, 3-month futures, 5-month futures, and 7-month futures, respectively.



Using Ito's lemma and the martingale property of futures prices, we have

$$\begin{aligned}
& dG_i(t,T) \\
= & \sum_{j=1}^{n} c_{S_j}(t,T)G_i(t,T)\sigma_{S_j}dW_{S_j}(t) + \sum_{j=1}^{n} c_{\delta_j}(t,T)G_i(t,T)\sigma_{\delta_j}dW_{\delta_j}(t) \\
= & \sum_{j=1}^{n}\left(\sigma_{S_j}\theta_{S_j0}c_{S_j}(t,T)G_i(t,T) + \sigma_{\delta_j}\theta_{\delta_j0}c_{\delta_j}(t,T)G_i(t,T)\right)dt \\
& + \sum_{j=1}^{n}\sigma_{S_j}c_{S_j}(t,T)G_i(t,T)dW^P_{S_j}(t) + \sum_{j=1}^{n}\sigma_{\delta_j}c_{\delta_j}(t,T)G_i(t,T)dW^P_{\delta_j}(t).
\end{aligned}$$

This equation states two facts. First, the drift term under the risk-neutral probability includes an error term equal to 0. Second, the drift term under the natural probability is nonlinearly affected by futures prices $G_j(t,T)$. This means that in either case, the adjustment coefficients for futures prices are different from the coefficients of linear relations a_i and adjustment coefficients b_i for spot prices.

We emphasize that the linear relation is not observable in the GSC model. There are two aspects of this unobservability. First, it is modeled as spot prices, which are not observable. If we model the linear relation using futures prices, the advantage of our model will be the observability of the price, which allows us to use simple regression analysis and avoid using the more technical Kalman filter. Second, we modeled the linear relation under the risk-neutral probability, which is not observable from the historical data. While a_i do not change with changes in probabilities, b_i do, as we have seen in the equation above. The adjustment coefficients are changed by the market price of risks; this implies that if cointegration exists, the effects of the error correction term on spot prices under the natural probability and under the risk-neutral probability will be different. Thus, it may be interesting to model the linear relations among observable futures prices under the natural probability instead of unobservable spot prices under the risk-neutral probability, and analyze the effects on spot prices and other derivatives.

5.2 Multidimensional $z(t)$

In this paper, we have assumed that there is only one linear relation, which is represented by the term $z(t)$. This can be relaxed to $h(< n)$ different linear relations $[z_1(t) \dots z_h(t)]^\top$ that can be formalized as

$$\begin{aligned} d\ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) + \sum_{j=1}^{h} b_{ij} z_j(t)\right) dt + \sigma_{S_i} dW_{S_i}(t), \quad i = 1, \dots, n \\ d\delta_i(t) &= \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t), \quad i = 1, \dots, n \\ z_j(t) &= \mu_z + a_{0j} t + \sum_{i=1}^{n} a_{ij} \ln S_i(t) \quad j = 1, \dots, h. \end{aligned}$$

It is then simple to derive the futures and call option formulae. We can also extend the assumption on market price of risk and formalize the state and observation equations for the Kalman filters. The difficulty of this model stems from the number of parameters to consider when estimating the model. The parameters to be estimated are $n(1+2n)$ parameters for volatilities and correlations, $2n$ parameters for convenience yields $(\hat{\alpha}, \kappa)$, $2h(n+1)$ parameters for linear relations $(mu_{zj}, a_{0j}, a_{ij}, b_{ij})$, $2n$ parameters for the market price of risks $(\boldsymbol{\theta})$, and other parameters that depend on the number of commodities and futures maturity data used for covariance matrix $\boldsymbol{R}$ in the observation equation. If we assume three commodities and two linear relations for the model using three maturities of futures for each commodity, there will be 55 parameters to be estimated. To conduct a realistic empirical analysis, the numbers of commodities and linear relations used have to be much smaller.

Furthermore, we can incorporate seasonality into the model. There are various ways of modeling seasonality.[15] One suggestion is the following.

$$\begin{aligned} d\ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) + b_i z(t)\right) dt + \sigma_{S_i} dW_{S_i}(t) \\ &\quad + \left(\sum_{m_i=1}^{M_i} \phi_{i,m_i,1} \cos(2\pi m_i t) + \phi_{i,m_i,2} \sin(2\pi m_i t)\right) dt + \sigma_{S_i} dW_{S_i}(t) \\ d\delta_i(t) &= \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t), \quad i = 1, \dots, n \\ z(t) &= \mu_z + a_0 t + \sum_{i=1}^{n} a_i \ln S_i(t). \end{aligned}$$

[15] Other models that include seasonality in commodity spot prices are Hannan, Terrell, and Tuckwell (1970), Manoliu and Tompaidis (2002), Richter and Sorensen (2002), Sorensen (2002), Geman and Nguyen (2005), Cortazar et al. (2008), Paschke and Prokopczuk (2009), and Casassus et al. (2009).

In this model, the seasonality is in the drift term of the dynamics of the logarithm of commodity prices $d \ln S_i(t)$. This can be interpreted as demeaned seasonality in the logarithm of commodity prices. Consider the dynamics of the logarithm of commodity prices without the $z(t)$ term. Integrating

$$\begin{aligned} d \ln S_i(t) &= \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) \right) dt \\ &+ \left(\sum_{m_i=1}^{M_i} \phi_{i,m_i,1} \cos(2\pi m_i t) + \phi_{i,m_i,2} \sin(2\pi m_i t) \right) dt + \sigma_{S_i} dW_{S_i}(t) \end{aligned}$$

from 0 to t, we have

$$\begin{aligned} \ln S_i(t) &= \ln S_i(0) + \int_0^t \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) \right) dt \\ &+ \sum_{m_i=1}^{M_i} \frac{\phi_{i,m_i,1}}{2\pi m_i} \sin(2\pi m_i t) - \frac{\phi_{i,m_i,2}}{2\pi m_i} \sin(2\pi m_i t) - \sum_{m_i=1}^{M_i} \frac{\phi_{i,m_i,2}}{2\pi m_i} \\ &+ \sigma_{S_i} W_{S_i}(t). \end{aligned}$$

This implies that the above model includes the demeaned seasonality in the logarithm of commodity prices.

6 Conclusion

In this paper, we formulated a commodity pricing model that incorporates the effect of linear relations among commodity prices, which includes cointegration under certain conditions. We derived futures and call option pricing formulae and showed that, in contrast to Duan and Pliska (2004), the linear relations among commodity prices, or the error correction term under appropriate conditions, should affect these derivative prices in the standard setup of commodity pricing. Furthermore, we derived the condition for the model to be cointegrated.

We emphasized that the proposed model can be interpreted as a generalization of standard commodity models, especially the GS model. This is because we decomposed the deviation of the drift in commodity returns from the risk-free rate under the risk-neutral probability into two components: convenience yield and the linear relation term $z(t)$. The proposed model can thus describe not only the usual storage effects captured by the convenience yield, but also other causes such as impacts from other commodity prices and transaction costs.

In the empirical analysis, we assumed that the market price of risk is linear in the convenience yield and the term $z(t)$, and utilized the Kalman filter technique. Using crude oil and heating oil market data, we estimated the proposed model. The results suggested that the linear relations among commodity prices affect their derivative prices empirically. We also implemented the model to examine the hedging of long-term futures.

Finally, it should be noted that while the linear relations among spot prices play an important role, such spot prices are assumed to be unobservable in standard commodity pricing models, including ours. Thus, it would be interesting to model the linear relations among observable futures prices instead of unobservable spot prices, and analyze the effects of the linear relation, or cointegration under certain conditions, on derivatives.

It should be also noted that, as Duan and Pliska (2004) showed, if the volatilities of commodity returns are stochastic, then cointegration affects derivative prices. Although they do not investigate the effect of linear relations among spot prices on derivative prices, Trolle and Schwartz (2009) developed a commodity derivative pricing model with stochastic volatility. Hence, it would also be interesting to advance a commodity derivative pricing model to incorporate linear relations among spot prices under stochastic volatility of their returns. We leave these questions for future study.

Appendices

Appendix 1 Proof of Proposition 2.2

We prove the call option pricing formula. From Harrison and Kreps (1979) or Harrison and Pliska (1981), we have

$$\begin{aligned} C_i(t,T) &= e^{-r(T-t)} E_t[(S_i(T)-K)^+] \\ &= e^{-r(T-t)} \int_D (e^{x_i}-K) n(x_i|\mu_{X_i}(t,T), \sigma^2_{X_i}(t,T)) dx_i, \end{aligned}$$

where $n(x|\mu,\sigma^2)$ is the density function of the normal distribution with mean μ and variance σ^2, and

$$D = \{x_i | x_i \geq \ln K\}.$$

The integral can be calculated as

$$\int_D \exp\{x_i\} n(x_i|\mu_{X_i}, \sigma^2_{X_i}) dx_i = \exp\left\{\mu_{X_i} + \frac{\sigma^2_{X_i}}{2}\right\} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_{i1}} \exp\left\{-\frac{y^2}{2}\right\} dy$$

where

$$d_{i1} = \frac{-\ln K + \mu_{X_i} + \sigma^2_{X_i}}{\sigma_{X_i}},$$

and we omit the time parameters such as $\mu_{X_i} = \mu_{X_i}(t,T)$ for notational convenience. Furthermore,

$$\int_D \frac{1}{\sqrt{2\pi}\sigma_{X_i}} \exp\left\{-\frac{(x_i-\mu_{X_i})^2}{2\sigma^2_{X_i}}\right\} dx_i = \int_{-\infty}^{d_{i2}} \frac{1}{\sqrt{2\pi}} \exp\left\{-\frac{y^2}{2}\right\} dy,$$

where

$$d_{i2} = \frac{-\ln K + \mu_{X_i}}{\sigma_{X_i}},$$

and again we omit the time parameters. Collecting all terms, we have

$$C_i(t,T) = e^{-r(T-t)+\mu_{X_i}(t,T)+\frac{\sigma^2_{X_i}(t,T)}{2}} \Phi(d_{i1}(t,T)) - Ke^{-r(T-t)} \Phi(d_{i2}(t,T)).$$

Appendix 2 Derivation of Spot and Futures Commodity Prices

In this subsection, we derive the closed formula for $S_i(T)$ and futures price $G_i(t,T)$ without integrals and matrix forms for the case when $b \neq 0$.[16] We assume that $\kappa_i \neq 0, \forall i$. Furthermore, for $b \neq 0$, we assume that $b - \kappa_i \neq 0$, $b + \kappa_i \neq 0, \forall i$ and $\kappa_i + \kappa_j \neq 0, \forall i, j$.

First, note that equation (1) is equivalent to

$$\begin{aligned} S_i(T) &= S_i(t) \exp\{\hat{X}_i(t,T)\} \\ \hat{X}_i(t,T) &= \int_t^T \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(s) + b_i z(s) \right) ds + \int_t^T \sigma_{S_i} dW_{S_i}(s). \end{aligned}$$

The key point of the derivation is the calculation of the term $z(t)$ as follows:

$$\begin{aligned} \int_t^T z(s) ds &= \frac{1}{b}(z(T) - z(t)) + m(T-t) + \sum_{j=1}^n \int_t^T \frac{a_j}{b} \delta_j(s) ds \\ &\quad - \sum_{i=1}^n \frac{a_i}{b} \sigma_{S_i} (W_{S_i}(T) - W_{S_i}(t)) \end{aligned}$$

and

$$\begin{aligned} z(T) &= e^{b(T-t)} z(t) + \int_t^T e^{b(T-s)} (-bm - \sum_{i=1}^n a_i \delta_i(s)) ds \\ &\quad + \sum_{i=1}^n \int_t^T e^{b(T-s)} a_j \sigma_{S_j} dW_{S_i}(s) \\ &= e^{b(T-t)} z(t) + m(1 - e^{b(T-t)}) - \int_t^T e^{b(T-s)} \sum_{i=1}^n a_i \delta_i(s)) ds \\ &\quad + \sum_{j=1}^n \int_t^T e^{b(T-s)} a_j \sigma_{S_j} dW_{S_i}(s). \end{aligned}$$

Hence, we have

$$\hat{X}_i(t,T) \triangleq \int_t^T \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(s) + b_i z(s) \right) ds + \int_t^T \sigma_{S_i} dW_{S_i}(s)$$

[16] For other cases, including the case when $b = 0$, the proofs can be obtained from the author on request.

$$
\begin{aligned}
= \quad & \left(r + b_i m - \frac{\sigma_i^2}{2} - \hat{\alpha}_i + \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b}\right)(T-t) \\
& + \frac{b_i(m - z(t))}{b}(1 - e^{b(T-t)}) - \sum_{j=1}^{n} \frac{b_i a_j \delta_j(t)}{b(b+\kappa_j)}(e^{b(T-t)} - e^{-\kappa_j(T-t)}) \\
& - \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b^2}(e^{b(T-t)} - 1) + \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b(b+\kappa_j)}(e^{b(T-t)} - e^{-\kappa_j(T-t)}) \\
& + \frac{(\hat{\alpha}_i - \delta_i(t))}{\kappa_i}(1 - e^{-\kappa_i(T-t)}) - \sum_{j=1}^{n} \frac{b_i a_j(\hat{\alpha}_j - \delta_j(t))}{b\kappa_j}(1 - e^{-\kappa_j(T-t)}) \\
& + \sigma_{S_i}(W_{S_i}(T) - W_{S_i}(t)) - \frac{1}{\kappa_i}\sigma_{\delta_i}(W_{\delta_i}(T) - W_{\delta_i}(t)) \\
& - \sum_{j=1}^{n} \frac{b_i a_j}{b}\sigma_{S_j}(W_{S_j}(T) - W_{S_j}(t)) + \sum_{j=1}^{n} \frac{b_i a_j}{b\kappa_j}\sigma_{\delta_j}(W_{S_i}(T) - W_{S_i}(t)) \\
& + \sum_{j=1}^{n} \frac{b_i a_j}{b} \int_t^T e^{b(T-s)}\sigma_{S_j} dW_{S_j}(s) \\
& + \int_t^T \frac{e^{-\kappa_i(T-s)}}{\kappa_i}\sigma_{\delta_i} dW_{\delta_i}(s) - \sum_{j=1}^{n} \int_t^T \frac{b_i a_j e^{-\kappa_j(T-s)}}{b\kappa_j}\sigma_{\delta_j} dW_{\delta_j}(s) \\
& - \sum_{j=1}^{n} \int_t^T \frac{b_i a_j}{b(b+\kappa_j)}(e^{b(T-s)} - e^{-\kappa_j(T-s)})\sigma_{\delta_j} dW_{\delta_j}(s).
\end{aligned}
$$

The $\mu_{\hat{X}_i}(t,T)$ and $\sigma_{\hat{X}_i}(t,T)$ are

$$
\mu_{\hat{X}_i}(t,T) \quad = \quad E_t[\hat{X}_i(t,T)]
$$

$$
\begin{aligned}
&= \left(r + b_i m - \frac{\sigma_{S_i}^2}{2} - \hat{\alpha}_i + \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b}\right)(T-t) \\
&\quad + \frac{b_i(m - z(t))}{b}(1 - e^{b(T-t)}) - \sum_{j=1}^{n} \frac{b_i a_j \delta_j(t)}{b(b+\kappa_j)}(e^{b(T-t)} - e^{-\kappa_j(T-t)}) \\
&\quad - \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b^2}(e^{b(T-t)} - 1) + \sum_{j=1}^{n} \frac{b_i a_j \hat{\alpha}_j}{b(b+\kappa_i)}(e^{b(T-t)} - e^{-\kappa_j(T-t)}) \\
&\quad + \frac{(\hat{\alpha}_i - \delta_i(t))}{\kappa_i}(1 - e^{-\kappa_i(T-t)}) - \sum_{j=1}^{n} \frac{b_i a_j(\hat{\alpha}_j - \delta_j(t))}{b\kappa_j}(1 - e^{-\kappa_j(T-t)})
\end{aligned}
$$

and

$$
\begin{aligned}
\sigma_{\hat{X}_i}^2(t,T) &= E_t[(\hat{X}_i(t,T) - \mu_{\hat{X}_i}(t,T))^2] \\
&= \left(\sigma_{S_i}^2 + \frac{\sigma_{\delta_i}^2}{\kappa_i^2} - \frac{2\sigma_{S_i\delta_i}}{\kappa_i} - \sum_{j=1}^{n} \frac{2b_i a_j}{b\kappa_i\kappa_j}\sigma_{\delta_i\delta_j}\right. \\
&\quad + \sum_{j=1}^{n} \frac{2b_i a_j \sigma_{S_j\delta_i}}{b\kappa_i} + \sum_{j,k=1}^{n} \frac{b_i^2 a_j a_k \sigma_{\delta_j\delta_k}}{b^2\kappa_i\kappa_k} + \sum_{j=1}^{n} \frac{2b_i a_j \sigma_{S_i\delta_j}}{b\kappa_j} \\
&\quad \left. - \sum_{j,k=1}^{n} \frac{2b_i^2 a_j a_k \sigma_{S_j\delta_k}}{b^2\kappa_k} - \sum_{j=1}^{n} \frac{2b_i a_j \sigma_{S_iS_j}}{b} + \sum_{j,k=1}^{n} \frac{b_i^2 a_j a_k \sigma_{S_jS_k}}{b^2}\right)(T-t) \\
&\quad + \frac{\sigma_{\delta_i}^2}{2\kappa_i^3}(1 - e^{-2\kappa_i(T-t)}) - \sum_{j=1}^{n} \frac{2b_i a_j \sigma_{\delta_i\delta_j}}{b\kappa_i\kappa_j(\kappa_i+\kappa_j)}(1 - e^{-(\kappa_i+\kappa_j)(T-t)}) \\
&\quad + \sum_{j,k=1}^{n} \frac{b_i^2 a_j a_k \sigma_{\delta_j\delta_k}}{b^2\kappa_j\kappa_k(\kappa_j+\kappa_k)}(1 - e^{-(\kappa_j+\kappa_k)(T-t)}) \\
&\quad - \frac{1}{\kappa_j - b}(1 - e^{-(\kappa_j-b)(T-t)}) - \frac{1}{\kappa_k - b}(1 - e^{-(\kappa_k-b)(T-t)}) \\
&\quad \left. + \frac{1}{\kappa_j+\kappa_k}(1 - e^{-(\kappa_j+\kappa_k)(T-t)})\right\}
\end{aligned}
$$

$$
\begin{aligned}
&-\sum_{j,k=1}^{n}\frac{b_i^2 a_j a_k \sigma_{S_j S_k}}{2b^3}(1-e^{2b(T-t)})\\
&-\sum_{j,k=1}^{n}\frac{2b_i^2 a_j a_k \sigma_{S_k \delta_j}}{b^2(b+\kappa_j)}\left\{\frac{-1}{2b}(1-e^{2b(T-t)})-\frac{1}{\kappa_j-b}(1-e^{-(\kappa_j-b)(T-t)})\right\}\\
&+2\left(-\frac{\sigma_{\delta_i}^2}{\kappa_i^3}+\frac{\sigma_{S_i\delta_i}}{\kappa_i^2}+\sum_{j=1}^{n}\frac{b_i a_j \sigma_{\delta_i\delta_j}}{b\kappa_i^2\kappa_j}-\sum_{j=1}^{n}\frac{b_i a_j \sigma_{S_j\delta_i}}{b\kappa_i^2}\right)(1-e^{-\kappa_i(T-t)})\\
&+\sum_{j=1}^{n}2\left(\frac{b_i a_j \sigma_{\delta_i\delta_j}}{b\kappa_i\kappa_j^2}-\frac{b_i a_j \sigma_{S_i\delta_j}}{b\kappa_j^2}-\sum_{k=1}^{n}\frac{b_i^2 a_j a_k \sigma_{\delta_j\delta_k}}{b^2\kappa_j^2\kappa_k}+\sum_{k=1}^{n}\frac{b_i^2 a_j a_k \sigma_{S_k\delta_j}}{b^2\kappa_j^2}\right)\\
&\times(1-e^{-\kappa_j(T-t)})\\
&+\sum_{j=1}^{n}2\left(\frac{b_i a_j \sigma_{\delta_i\delta_j}}{b\kappa_i(b+\kappa_j)}-\frac{b_i a_j \sigma_{S_i\delta_j}}{b(b+\kappa_j)}-\sum_{k=1}^{n}\frac{b_i^2 a_j a_k \sigma_{\delta_j\delta_k}}{b^2\kappa_k(b+\kappa_j)}\right.\\
&\left.+\sum_{k=1}^{n}\frac{b_i^2 a_j a_k \sigma_{S_k\delta_j}}{b^2(b+\kappa_j)}\right)\left\{\frac{-1}{b}(1-e^{b(T-t)})-\frac{1}{\kappa_j}(1-e^{-\kappa_j(T-t)})\right\}\\
&+2\left(\sum_{j=1}^{n}\frac{b_i a_j \sigma_{S_j\delta_j}}{\kappa_i b^2}-\sum_{j,k=1}^{n}\frac{b_i^2 a_j a_k \sigma_{S_j\delta_k}}{\kappa_k b^3}-\sum_{j=1}^{n}\frac{b_i a_j \sigma_{S_i S_j}}{b^2}+\sum_{j,k=1}^{n}\frac{b_i^2 a_j a_k \sigma_{S_j S_k}}{b^3}\right)\\
&\times(1-e^{b(T-t)})\\
&-\sum_{j=1}^{n}\frac{2b_i a_j \sigma_{\delta_i\delta_j}}{b\kappa_i(b+\kappa_j)}\left\{\frac{1}{\kappa_i-b}(1-e^{-(\kappa_i-b)(T-t)})-\frac{1}{\kappa_i+\kappa_j}(1-e^{-(\kappa_i+\kappa_j)(T-t)})\right\}\\
&+\sum_{j=1}^{n}\frac{2b_i a_j \sigma_{S_j\delta_i}}{b\kappa_i(\kappa_i-b)}(1-e^{-(\kappa_i-b)(T-t)})\\
&+\sum_{j,k=1}^{n}\frac{2b_i^2 a_j a_k \sigma_{\delta_j\delta_k}}{b^2\kappa_k(b+\kappa_j)}\left\{\frac{1}{\kappa_k-b}(1-e^{-(\kappa_k-b)(T-t)})-\frac{1}{\kappa_j+\kappa_k}(1-e^{-(\kappa_j+\kappa_k)(T-t)})\right\}\\
&-\sum_{j,k=1}^{n}\frac{2b_i^2 a_j a_k \sigma_{S_j\delta_k}}{b^2\kappa_k(\kappa_k-b)}(1-e^{-(\kappa_k-b)(T-t)}).
\end{aligned}
$$

We have the following proposition, which shows the price formula for futures.

Proposition A.1. *Assuming (1), (2), and (3), the futures price of commod-*

ity i with maturity T at t is given by

$$\begin{aligned} G_i(t,T) &= E_t[S_i(T)] \\ &= S_i(t)\exp\left\{\mu_{\hat{X}_i}(t,T) + \frac{\sigma^2_{\hat{X}_i}(t,T)}{2},\right\} \end{aligned}$$

where $\mu_{\hat{X}_i}(t,T) = E_t[\hat{X}_i(t,T)]$ *and* $\sigma^2_{\hat{X}_i}(t,T) = E_t[(\hat{X}_i(t,T) - \mu_{\hat{X}_i}(t,T))^2]$.

Proof. Using risk neutrality and given the properties of the moment generating function, we obtain the futures price of commodity i. □

Appendix 3 Cointegration Condition for the GSC model

In this subsection, we provide the cointegration condition for the GSC model. Recall that the definition of cointegration is that $\ln S_i(t) - \ln S_i(t - \triangle t)$ is $I(0)$ for every i and $\sum_{i=1}^{n} a_i \ln S_i(t)$ is stationary.

We use the following proposition, which enhances a proposition from Hamilton (1994).[17]

Proposition A.2. *Let $\boldsymbol{x}(t)$ be a vector satisfying*

$$\boldsymbol{x}(t) = \boldsymbol{\mu}_x + \sum_{s=0}^{\infty} \boldsymbol{\Phi}(s)\boldsymbol{\varepsilon}(t-s),$$

where $\boldsymbol{\varepsilon}(t)$ is a zero mean covariance-stationary process, i.e. $E[\boldsymbol{\varepsilon}(t)] = \boldsymbol{0}$, $E[\boldsymbol{\varepsilon}(t)\boldsymbol{\varepsilon}(t-s)^\top] = \boldsymbol{\Omega}(s)$ and $\{\boldsymbol{\Phi}(t)\}$ is absolutely summable, i.e. $\sum_{s=0}^{\infty} |\phi(s)_{ij}| < \infty$, where $\phi_{ij}(t)$ is the row i, column j element of $\boldsymbol{\Phi}(t)$. Then the autocovariance is

$$E[(\boldsymbol{x}(t) - \boldsymbol{\mu}_x)(\boldsymbol{x}(t-s) - \boldsymbol{\mu}_x)^\top] = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \boldsymbol{\Phi}(u)\boldsymbol{\Omega}(-u+s+v)\boldsymbol{\Phi}^\top(v),$$

which implies that the autocovariance is only a function of lag s.

Proof. The proof follows the proposition shown in Hamilton (1994). First, define

$$y_{il}(t) = \sum_{u=0}^{\infty} \phi_{il}(u)\varepsilon_l(t-u),$$

[17] Cf. Hamilton(1994), Proposition 10.2, p. 263.

where $\phi_{il}(u)$ is the row i, column l element of matrix $\mathbf{\Phi}(u)$. Note that

$$x_i(t) = \mu_{xi} + \sum_{l=1}^{n} y_{il}(t),$$

where $x_i(t)$ and μ_{xi} are the ith elements of $\boldsymbol{x}(t)$ and $\boldsymbol{\mu}_x$, respectively. Let us calculate the autocovariance of $y_{il}(t)$.

$$E[y_{il}(t)y_{jm}(t-s)] = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty} \phi_{il}(u)\phi_{jm}(v)\omega_{lm}(-u+s+v),$$

where we denote $\omega_{lm}(t)$ as the row l, column m element of $\Omega(t)$ and interchange the expectation operator and summation operator because

$$\sum_{u=0}^{\infty}\sum_{v=0}^{\infty} |\phi_{il}(u)\phi_{jm}(v)| = \sum_{u=0}^{\infty} |\phi_{il}(u)| \cdot \sum_{v=0}^{\infty} |\phi_{jm}(v)| < \infty.$$

We calculate the autocovariance of $\boldsymbol{x}(t)$.

$$E[(x_i(t)-\mu_{xi})(x_j(t-s)-\mu_{xj})] = \sum_{u=0}^{\infty}\sum_{v=0}^{\infty}\sum_{l=1}^{n}\sum_{m=1}^{n} \phi_{il}(u)\phi_{jm}(v)\omega_{lm}(-u+s+v)$$

As $\sum_{l=1}^{n}\sum_{m=1}^{n} \phi_{il}(u)\phi_{jm}(v)\omega_{lm}(-u+s+v)$ is the row i, column j element of $\mathbf{\Phi}(u)\mathbf{\Omega}(-u+s+v)\mathbf{\Phi}(v)$, the proposition is proved. $\square$

Let us assume that $e^{b\triangle t} < 1(\iff b < 0), e^{-\kappa_i \triangle t} < 1(\iff \kappa_i > 0)$.

We now prove that $\ln S_i(t)$ is cointegrated. First, we see that $z(t)$ is stationary. Note that

$$\delta_i(t) = e^{-\kappa_i \triangle t}\delta_i(t-\triangle t) + \int_{t-\triangle t}^{t} e^{-\kappa_i(t-s)}\kappa_i\hat{\alpha}_i ds + \int_{t-\triangle t}^{t} e^{-\kappa_i(t-s)}\sigma_{\delta_i} dW_{\delta_i}(s).$$

As $e^{-\kappa_i \triangle t} < 1$, this yields

$$\delta_i(t) \quad = \quad \sum_{s=0}^{\infty} e^{-s\kappa_i \triangle t}(d_{\delta_i}(t-s\triangle t) + \varepsilon_{\delta_i}(t-s\triangle t)),$$

where

$$\begin{aligned} d_{\delta_i}(t) &= \int_{t-\triangle t}^{t} e^{-\kappa_i(t-s)}\kappa_i\hat{\alpha}_i ds \\ \varepsilon_{\delta_i}(t) &= \int_{t-\triangle t}^{t} e^{-\kappa_i(t-s)}\sigma_{\delta_i} dW_{\delta_i}(s). \end{aligned}$$

Note that $d_{\delta_i}(t)$ does not depend on t, which can be easily confirmed by integrating or using changes of variables.

The $z(t)$ can be expanded as follows.

$$
\begin{aligned}
z(t) &= e^{b\triangle t}z(t-\triangle t) + \int_{t-\triangle t}^{t} e^{b(t-s)}\left(\sum_{i=1}^{n} a_i r - \frac{1}{2}\sum_{i=1}^{n} a_i \sigma_{S_i}^2\right) ds \\
&\quad - \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)}\delta_i(s) ds \\
&\quad + \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)}\sigma_{S_i} dW_{S_i}(s)
\end{aligned}
$$

$$
\begin{aligned}
&= e^{b\triangle t}z(t-\triangle t) + \int_{t-\triangle t}^{t} e^{b(t-s)}\left(a_0 + \sum_{i=1}^{n} a_i r - \frac{1}{2}\sum_{i=1}^{n} a_i \sigma_{S_i}^2\right) ds \\
&\quad - \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)-\kappa_i(s-t+\triangle t)} ds \\
&\quad \times \left(\sum_{v=1}^{\infty} e^{-(v-1)\kappa_i \triangle t}\left(d_{\delta_i}(t-v\triangle t) + \varepsilon_{\delta_i}(t-v\triangle t)\right)\right) \\
&\quad - \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)}\hat{\alpha}_i - e^{b(t-s)-\kappa_i(s-t+\triangle t)}\hat{\alpha}_i ds \\
&\quad + \int_{t-\triangle t}^{t}\int_{u}^{t} e^{b(t-s)-\kappa_i(s-u)}\sigma_{\delta_i} ds dW_{\delta_i}(u) \\
&\quad + \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)}\sigma_{S_i} dW_{S_i}(s),
\end{aligned}
$$

where we use Fubini's theorem for stochastic integrals and

$$
\begin{aligned}
&\int_{t-\triangle t}^{t} e^{b(t-s)}\delta_i(s) ds \\
= &\int_{t-\triangle t}^{t} e^{b(t-s)-\kappa_i(s-t+\triangle t)}\delta_i(t-\triangle t) + e^{b(t-s)}\hat{\alpha}_i - e^{b(t-s)-\kappa_i(s-t+\triangle t)}\hat{\alpha}_i ds \\
&+ \int_{t-\triangle t}^{t}\int_{u}^{t} e^{b(t-s)-\kappa_i(s-u)}\sigma_{\delta_i} ds dW_{\delta_i}(u).
\end{aligned}
$$

Thus, from $e^{b\triangle t} < 1$,

$$
\begin{aligned}
z(t) &= \sum_{s=0}^{\infty} e^{sb\triangle t}(d_z(t - s\triangle t) + \varepsilon_{z1}(t - s\triangle t)) \\
&\quad + \sum_{s=0}^{\infty}\sum_{v=0}^{\infty} e^{sb\triangle t} e^{-v\kappa_i \triangle t} \varepsilon_{z2}(t - (s + v + 1)\triangle t),
\end{aligned}
$$

where

$$
\begin{aligned}
d_z(t) &= \int_{t-\triangle t}^{t} e^{b(t-s)} \left(\sum_{i=1}^{n} a_i r - \frac{1}{2} \sum_{i=1}^{n} a_i \sigma_{S_i}^2 \right) ds \\
&\quad - \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)-\kappa_i(s-t+\triangle t)} ds \sum_{v=1}^{\infty} e^{-(v-1)\kappa_i \triangle t} d_{\delta_i}(t - v\triangle t) \\
&\quad - \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)} \hat{\alpha}_i - e^{b(t-s)-\kappa_i(s-t+\triangle t)} \hat{\alpha}_i ds \\
\varepsilon_{z1}(t) &= \int_{t-\triangle t}^{t} \int_{u}^{t} e^{b(t-s)-\kappa_i(s-u)} \sigma_{\delta_i} ds dW_{\delta_i}(u) \\
&\quad + \sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)} \sigma_{S_i} dW_{S_i}(s) \\
\varepsilon_{z2}(t) &= -\sum_{i=1}^{n} a_i \int_{t-\triangle t}^{t} e^{b(t-s)-\kappa_i(s-t+\triangle t)} ds \varepsilon_{\delta_i}(t).
\end{aligned}
$$

Again, $d_z(t)$ does not depend on t and $\varepsilon_{z1}(t)$ and $\varepsilon_{z2}(t)$ are white noise. From Proposition A.2, we can see that $z(t)$ is stationary. Therefore, $\ln S_i(t)$ is cointegrated.

Next, we check that $\ln S_i(t) - \ln S_i(t-\triangle t)$ is $I(0)$ for every i.

$$\begin{aligned}
& \ln S_i(t) - \ln S_i(t-\triangle t) \\
= & \int_{t-\triangle t}^{t} \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(s) + b_i z(s)\right) ds + \int_{t-\triangle t}^{t} \sigma_{S_i} dW_{S_i}(s) \\
= & \mu_{\ln S_i} + \sum_{s=0}^{\infty} \left(- \int_{t-\triangle t}^{t} e^{-\kappa_i(s-t+\triangle t)} ds \right) e^{-s\kappa_i \triangle t} \varepsilon_{\delta_i}(t-(s+1)\triangle t) \\
& + b_i e^{b\triangle t} \triangle t \left(\sum_{s=0}^{\infty} e^{sb\triangle t} \varepsilon_{z_1}(t-(s+1)\triangle t) \right. \\
& \left. + \sum_{s=0}^{\infty} \sum_{v=0}^{\infty} e^{sb\triangle t - v\kappa_i \triangle t} \varepsilon_{z_2}(t-(s+v+2)\triangle t) \right) \\
& - b_i \sum_{j=1}^{n} a_j \int_{t-\triangle t} \int_{t-\triangle t}^{s} e^{b(s-u)-\kappa_j(u-t+\triangle t)} du ds \\
& \times \left(\sum_{v=1}^{\infty} e^{-(v-1)\kappa_j \triangle t} \varepsilon_{\delta_j}(t-v\triangle t) \right) + \varepsilon_{\ln S_i}(t),
\end{aligned}$$

where

$$\begin{aligned}
\mu_{\ln S_i} = & \left(r - \frac{\sigma_{S_i}^2}{2} \right) \triangle t - \hat{\alpha}_i \triangle t + \hat{\alpha}_i \int_{t-\triangle t}^{t} e^{-\kappa_i(s-t+\triangle t)} ds \\
& + b_i \left(\sum_{j=1}^{n} a_j r - \frac{\sum_{j=1}^{n} a_j \sigma_{S_j}^2}{2} \right) \int_{t-\triangle t}^{t} \int_{t-\triangle t}^{s} e^{b(s-u)} du ds \\
& - \sum_{j=1}^{n} a_j \hat{\alpha}_j \int_{t-\triangle t}^{t} \int_{t-\triangle t}^{s} e^{b(s-u)} (1 - e^{-\kappa_j(u-t+\triangle t)}) du ds \\
& - \int_{t-\triangle t}^{t} e^{-\kappa_i(s-t+\triangle t)} ds \sum_{s=0}^{\infty} e^{-s\kappa_i \triangle t} (d_{\delta_i}(t-(s+1)\triangle t))
\end{aligned}$$

$$
\begin{aligned}
& +b_i e^{b\triangle t}\triangle t\left(\sum_{s=0}^{\infty} e^{sb\triangle t} d_z((s+1)\triangle t)\right) \\
& -b_i \sum_{j=1}^{n} a_j \int_{t-\triangle t}^{t}\int_{t-\triangle t}^{s} e^{b(s-u)-\kappa_j(u-t+\triangle t)} duds \\
& \times\left(\sum_{v=1}^{\infty} e^{-(v-1)\kappa_j\triangle t} d_{\delta_j}(t-v\triangle t)\right) \\
\varepsilon_{\ln S_i} = & \int_{t-\triangle t}^{t}\int_{v}^{t}\int_{u}^{s} e^{b(s-u)-\kappa_i(u-v)}\sigma_{\delta_i} dudsdW_{\delta_i}(v) \\
& +\sum_{j=1}^{n} a_j \int_{t-\triangle t}^{t}\int_{u}^{t} e^{b(s-u)}\sigma_{S_i} dsdW_{S_j}(u) \\
& +\int_{t-\triangle t}^{t} \sigma_{S_i} dW_{S_i}(s).
\end{aligned}
$$

$\mu_{\ln S_i}$ does not depend on time t by using changes of variables. Furthermore, note that the stochastic terms are white noise. Therefore, we can use Proposition A.2, which concludes that $\ln S_i(t) - \ln S_i(t-\triangle t)$ are $I(0)$. This completes the proof.

Appendix 4 Other Empirical Results for the GSC Model

In this subsection, we show the empirical results for the other cases: (i) $b \neq 0$ and $a_1 = 1$, (iii) $b = 0, a_1 = 1, b_1 = -a_2 b_2$, (iv) $b = 0, a_1 = 0, a_2 = 0$, and (v) $b = 0, a_1 = 0, b_2 = 0$. The estimation results are presented in Tables 6 and 7. Except for case (iii), we see that the AIC is lower than that in the GS model. Furthermore, we analyze the result of case (iv), because this case has the lowest AIC, including case (i) $b \neq 0$ and $a_1 = 1$ and the GS model. Recall that this case does not satisfy the condition of cointegration and, thus, the estimated parameters are not valid, which means these are not comparable to standard deviations.

For case (iv), in which the linear relation vectors a_i are both 0, the adjustment speeds are $[b_1, b_2] = [-0.109099, 0.095277]$, respectively. The standard deviations for these parameters are very large. The time drift parameters a_0 and μ_z are 0.000001 and 0.012542, respectively.

Let us turn to the convenience yields. Note that κ_2 is negative, which means that the convenience yield is not stationary. Both long-term means

$\hat{\alpha}_i$ are small compared with the standard deviations, but the adjustment parameters κ_i are larger than the standard deviations.

A comparison of case (i) and the GS model does not reveal any significant differences between the volatility parameters. The differences we indicate are between the volatility parameters of the heating oil convenience yield σ_{δ_2}, correlations of heating oil price and crude oil convenience yield $\rho_{S_2\delta_1}$, and correlations of the convenience yields $\rho_{\delta_1\delta_2}$. σ_{δ_2} for case (iv) is much less volatile than the two previous models. The correlations $\rho_{S_2\delta_1}$ and $\rho_{\delta_1\delta_2}$ are lower.

Tables 8 and 9 show the root mean square error (RMSE) and mean error (ME) of the four cases. A comparison of the GS model and case (i) indicates that the result of heating oil for maturity 1 is not good, although maturity 5 is somewhat improved; there is no significant improvement or depreciation in the other parts.

Table 6: Estimated parameters, with standard errors in parentheses. Data are WTI and heating oil daily closing prices traded on the NYMEX from January 2, 1990, to July 30, 2010.

	(i)	(iii)
Volatility parameters		
σ_{S_1}	0.366667 (0.001942)	0.368996 (0.004328)
σ_{S_2}	0.403866 (0.002648)	0.356059 (0.005757)
σ_{δ_1}	0.292311 (0.001850)	0.198031 (0.003283)
σ_{δ_2}	0.689807 (0.007995)	0.211847 (0.005608)
$\rho_{S_1 S_2}$	0.714434 (0.005201)	0.858236 (0.007080)
$\rho_{S_1 \delta_1}$	0.737688 (0.004359)	0.002200 (0.020520)
$\rho_{S_1 \delta_2}$	0.000051 (0.012178)	0.000382 (0.056260)
$\rho_{S_2 \delta_1}$	0.528267 (0.006447)	-0.050049 (0.020261)
$\rho_{S_2 \delta_2}$	0.651368 (0.007394)	0.388430 (0.052287)
$\rho_{\delta_1 \delta_2}$	0.109110 (0.013473)	-0.000335 (0.001343)
Convenience yield parameters		
κ_1	1.142947 (0.007264)	0.886730 (0.005096)
κ_2	1.256849 (0.016329)	0.026644 (0.029419)
$\hat{\alpha}_1$	0.019111 (0.002216)	0.076056 (0.003200)
$\hat{\alpha}_2$	0.013057 (0.014289)	0.016721 (1.041906)
Linear relation parameters		
μ_z	0.023941 (0.045201)	-0.022223 (1.966758)
a_0	0.000106 (0.000003)	0.006951 (0.005982)
a_1	1.000000 (n.a.)	1.000000 (n.a.)
a_2	-0.725814 (0.007109)	-0.008302 (0.009216)
b_1	0.052334 (0.002303)	-0.000572 (n.a.)
b_2	0.366034 (0.005675)	-0.068889 (0.035397)
Market price of risk parameters		
$\theta_{S_1 0}$	0.199469 (0.222492)	0.082667 (0.330452)
$\theta_{S_2 0}$	-0.233062 (0.233117)	0.185452 (0.339923)
$\theta_{\delta_1 0}$	0.011876 (0.230454)	-1.799341 (0.332545)
$\theta_{\delta_2 0}$	0.006551 (0.273943)	-0.004155 (0.810891)
$R(1,1)$	0.000515 (0.000005)	0.000919 (0.000013)
$R(2,2)$	0.000000 (0.000000)	0.000030 (0.000001)
$R(3,3)$	0.000009 (0.000000)	0.000001 (0.000000)
$R(4,4)$	0.000000 (0.000000)	0.000002 (0.000000)
$R(5,5)$	0.000021 (0.000001)	0.000007 (0.000000)
$R(6,6)$	0.000001 (0.000001)	0.006362 (0.000168)
$R(7,7)$	0.001022 (0.000030)	0.000895 (0.000023)
$R(8,8)$	0.000696 (0.000022)	0.000015 (0.000001)
$R(9,9)$	0.000008 (0.000000)	0.000000 (0.000000)
$R(10,10)$	0.001017 (0.000028)	0.000766 (0.000018)
Log-likelihood	154178.537078	151160.610135
AIC	-308291.074155	-302257.220269
sample size	51590	51590

Table 7: Estimated parameters, with standard errors in parentheses. Data are WTI and heating oil daily closing prices traded on the NYMEX from January 2, 1990, to July 30, 2010.

	(iv)	(v)
Volatility parameters		
σ_{S_1}	0.350206 (0.001866)	0.336697 (0.001782)
σ_{S_2}	0.329622 (0.003536)	0.313910 (0.002938)
σ_{δ_1}	0.265981 (0.002176)	0.242048 (0.002008)
σ_{δ_2}	0.244615 (0.005482)	0.205013 (0.004326)
$\rho_{S_1 S_2}$	0.625170 (0.008841)	0.685699 (0.008048)
$\rho_{S_1 \delta_1}$	0.758072 (0.004341)	0.715504 (0.005356)
$\rho_{S_1 \delta_2}$	-0.000129 (0.011527)	-0.000452 (0.013543)
$\rho_{S_2 \delta_1}$	0.347681 (0.009722)	0.351016 (0.010844)
$\rho_{S_2 \delta_2}$	0.645149 (0.010296)	0.589559 (0.011140)
$\rho_{\delta_1 \delta_2}$	0.017574 (0.011490)	0.002437 (0.013464)
Convenience yield parameters		
κ_1	0.949412 (0.005239)	0.951637 (0.005549)
κ_2	-0.231644 (0.017156)	-0.345426 (0.019500)
$\hat{\alpha}_1$	0.016148 (0.051007)	0.007442 (0.043916)
$\hat{\alpha}_2$	0.014541 (0.191621)	0.149694 (0.009358)
Linear relation parameters		
μ_z	0.012542 (12.485066)	0.024759 (2.826261)
a_0	0.000001 (0.000519)	-0.000066 (0.006340)
a_1	0.000000 (n.a.)	0.000000 (n.a.)
a_2	0.000000 (n.a.)	-0.130021 (12.499299)
b_1	-0.109099 (108.382591)	0.030849 (2.963343)
b_2	0.095277 (94.836457)	0.000000 (n.a.)
Market price of risk parameters		
$\theta_{S_1 0}$	0.005576 (0.235339)	0.002583 (0.222497)
$\theta_{S_2 0}$	0.026724 (0.252499)	0.371485 (0.242683)
$\theta_{\delta_1 0}$	-0.093062 (0.248346)	-0.560535 (0.232429)
$\theta_{\delta_2 0}$	-0.000270 (0.380532)	0.000262 (0.383377)
$R(1,1)$	0.001041 (0.000016)	0.001124 (0.000018)
$R(2,2)$	0.000046 (0.000001)	0.000051 (0.000001)
$R(3,3)$	0.000000 (0.000000)	0.000000 (0.000000)
$R(4,4)$	0.000000 (0.000000)	0.000000 (0.000000)
$R(5,5)$	0.000009 (0.000000)	0.000008 (0.000000)
$R(6,6)$	0.006745 (0.000173)	0.006392 (0.000162)
$R(7,7)$	0.001100 (0.000028)	0.001091 (0.000029)
$R(8,8)$	0.000000 (0.000000)	0.000002 (0.000000)
$R(9,9)$	0.000000 (0.000000)	0.000002 (0.000000)
$R(10,10)$	0.001038 (0.000027)	0.001113 (0.000031)
Log-likelihood	154897.735778	154743.996275
AIC	-309731.471557	-309423.992550
sample size	51590	51590

Table 8: RMSE (root mean square error) and ME (mean error) for each futures contract.

Contracts	RMSE		ME	
Models	(ii)	(iii)	(ii)	(iii)
Crude oil				
Maturity 1	0.032943	0.038575	-0.002741	-0.002193
Maturity 3	0.020206	0.021061	-0.000003	0.000400
Maturity 5	0.018556	0.018369	0.000026	-0.000083
Maturity 7	0.017296	0.017350	-0.000030	-0.000495
Maturity 9	0.017152	0.016782	0.000170	-0.000185
Heating oil				
Maturity 1	0.024064	0.082284	0.000023	-0.000638
Maturity 3	0.037136	0.035022	-0.001603	0.000271
Maturity 5	0.031699	0.018464	-0.000226	0.000495
Maturity 7	0.017767	0.017380	-0.000012	0.000711
Maturity 9	0.036419	0.035860	-0.003012	0.001615

Table 9: RMSE (root mean square error) and ME (mean error) for each futures contract.

Contracts	RMSE		ME	
Models	(iv)	(v)	(iv)	(v)
Crude oil				
Maturity 1	0.039733	0.039573	-0.003794	-0.003158
Maturity 3	0.021257	0.021247	0.000062	0.000170
Maturity 5	0.018388	0.018378	0.000443	0.000248
Maturity 7	0.017312	0.017300	0.000412	0.000159
Maturity 9	0.016819	0.016791	0.000619	0.000544
Heating oil				
Maturity 1	0.084311	0.082893	-0.001628	-0.010059
Maturity 3	0.037614	0.037121	-0.000298	-0.003341
Maturity 5	0.018668	0.018706	0.000332	-0.000079
Maturity 7	0.017853	0.017942	0.000348	-0.000137
Maturity 9	0.037304	0.038042	-0.000148	-0.003425

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