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Analyzing Emission Allowance as a Derivative on Commodity-Spread

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Abstract

In this paper, we theoretically analyze price of emission allowance. Assuming that the value of emission allowance on the terminal day of a trading phase can be approximated by a spread of commodity prices (e.g. natural gas and coal prices) when the spread is positive and less than the penalty, we show that the emission allowance price is represented by the value of a portfolio of European call options on the spread of the commodities. Using this formula, we then obtain a hedging strategy for emission allowance trading. We also analyze option values embedded in emission allowance. By numerical analysis, we provide comparative statics on the property of emission allowance price and find that the embedded option values can be large.

Keywords: commodity prices, derivative, emission allowance, energy, hedging strategy.

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1 Introduction

Since the EU Emission Trading System (EU ETS) was launched in October 2003, the European Energy Exchange (EEX), European Climate Exchange (ECX), Powernext, Chicago Climate Exchange (CCX), and New York Mercantile Exchange (NYMEX) began trading EU allowances (EUAs), certified emission reductions (CERs), and their derivatives. The trading volumes in these markets are increasing as the EU ETS expands; therefore, the pricing of emission allowances is becoming an important issue.

In the academic literature, Cronshaw and Kruse (1996), Rubin (1996), Schennach (2000), Fehr and Hinz (2007), Seifert, Uhrig-Homberg, and Wagner (2008) investigated theoretically the prices of emission allowances. These studies focused on describing the univariate properties of emission allowance prices in terms of abatement costs, but did not examine the relations between prices of emission allowances and other commodities. In contrast, some empirical papers, such as Bunn and Fezzi (2007) and Mansanet-Bataller, Pardo, and Valor (2007), found relations between futures prices of commodities such as emission allowances, electricity, natural gas, and temperature. These results suggest the need for a model that incorporates price relations between emission allowances and other commodities.

Studies also exist on derivatives of emission allowances. Chao and Wilson (1993) assumed a fixed supply of and stochastic demand for allowances, and derived an explicit valuation formula for options on emission allowances. Maeda (2001) presented a forward pricing model of emission allowances with and without banking. Kijima, Maeda, and Nishide (2010) built a pricing model of emission allowances in a general equilibrium framework. Chesney and Taschini (2009) proposed a model under asymmetric information that allowed intertemporal banking and borrowing, and derived a closed-form pricing formula for a European option. Daskalakis, Psychoyois, and Markellos (2009) studied the spot and futures markets of emission allowances with several price processes including a mean reverting square-root process and jump process, and addressed the difference between interphase and intraphase markets. Again, these studies only considered the emission allowance price, and did not model explicitly the relations between commodity prices.

In this paper, we characterize the price of emission allowances by incorporating the interrelations between emission allowances and other commodities. More precisely, we focus on a situation where operational fuel switching is the main source of emission reductions and where the prices of emission allowances and fuel are related through optimal fuel switching by producers (e.g., power companies). We assume that there are two kinds of fuel (e.g., natural gas and coal[1]) by which one kind of good (e.g., electricity) is pro-

[1] Natural gas is more expensive but has a lower carbon content than coal.

duced with emissions of CO_2 as a by-product. We also assume that emission allowances are traded in a system similar to the EU ETS. That is, emission allowances are traded in a certain predetermined period, and compliance with the reductions is required at the end of period with a penalty for violation.

In this situation, the profit-maximization of producers leads to inter- and intratemporal conditions on prices of emission allowances and fuels. The former requires that the emission allowance price at any date should be equal to the present value of the emission allowance price at the end of the trading period. The latter requires that the marginal cost of fuel and emission allowance per unit of production should be equal across all kinds of fuels. Imposing these conditions, taking account of the penalty, and assuming that fuel prices follow Gibson–Schwartz-type (1990) stochastic processes, we provide a valuation formula for the emission allowance price in terms of a spread between fuel prices with the penalty.

It is worth noting that many other factors may affect the emission allowance price but are not incorporated in our analysis; such factors include demand for emission allowances from other industries, investment in abatement technologies, asymmetric information on emission reduction, uncertainty of institutional change, and so on. In reality, these factors can also be important determinants of emission prices. However, by focusing on the relation between operational fuel switching and the emission allowance price, we are able to characterize explicitly at least a part of the emission allowance price in terms of a spread between fuel prices with the penalty. In other words, we obtain an approximation of the emission allowance price that can be described by observable variables, i.e., prices of tradable commodities. This allows us to value emission allowances in part but in a tractable way.

This paper is organized as follows. In Section 2, we characterize the spot price of emission allowances as a derivative of a spread between fuel prices with the penalty. We also analyze the option values embedded in emission allowances and derive valuation formulae for futures and options on emission allowances. Using these valuation formulae, in Section 3, we characterize a hedging strategy of emission allowances using commodity futures. In Section 4, numerical analyses are provided. Section 5 concludes.

2 Prices of Emission Allowances and Their Derivatives

2.1 The Setup

Let us consider an economy that has a CO_2 emission trading system similar to the EU ETS. That is, the emission allowance of CO_2 is traded and its cumulative amount of emissions in period $[0, T]$ is required to be less than a certain limit. Excess emissions over the limit are penalized at the end of period T and emission allowances are traded throughout the period $[0, T]$. In this economy, we are interested in characterizing the price of emission allowances. For this purpose, we focus on the relation between the emission price and operational fuel switching.

To be more concrete, assume that there is a competitive power company that generates electricity by burning two kinds of fuel, such as natural gas and coal, while emitting CO_2 as a by-product. Assume for simplicity that this company is the dominant player in the economy and that its power-generating activities determine the relative prices of emission allowances, electricity, natural gas, and coal. Assume also that this power company already has adequate facilities to satisfy electricity demand by using either natural gas or coal as fuel, does not invest in new facilities in the period $[0, T]$, and decides which fuel to use to generate electricity depending on the prices of electricity, fuel, and emission allowances.

In this situation, it is well known that profit-maximization of the power company requires prices of electricity, fuel, and emission allowances to satisfy both intertemporal and intratemporal conditions. The former requires that the emission allowance price at the interim date $t(\leq T)$ should be equal to the present value of the emission allowance price at the end of period T.[2] The latter requires equality between marginal revenue of output (electricity) and marginal cost of input (fuel and emission allowances), which leads to an expression of the allowance price as a spread between fuel prices. In the following, we utilize these inter- and intratemporal conditions and characterize the price of emission allowances in terms of fuel prices.[3]

Denote by $S_e(t)$ the spot price of emission allowances at $t(\leq T)$ and by $S_e(T)$ the price at T. The intertemporal condition that the emission allowance price should satisfy is

$$S_e(t) \;=\; E_t[e^{-r(T-t)} S_e(T)] \tag{1}$$

[2] This is because emission allowances are needed only at the end date T when the central authority checks the companies in order to penalize any offenders. Thus, if the emission allowance price at $t(\leq T)$ is lower (resp. higher) than the present value of the price at T, the companies can increase their profits by adopting a trading strategy to buy (resp. sell) allowances at t and to sell (resp. buy) them back at T, which contradicts their profit maximization. See also Nakajima and Ohashi (2010).

[3] By making the strong assumption that these conditions apply to the emission allowance price such that it can be expressed in terms of fuel prices and the penalty and by ignoring other factors that may affect the emission allowance price, we are able to characterize explicitly the relation between the emission allowance price, fuel prices, and the penalty; this enables us to value the emission allowance in a tractable way. We assume that this model approximates the emission allowance price on some level.

where $E[\cdot]$ is expectation under risk-neutral probability P. Thus, to derive the emission allowance price $S_e(t)$, we need to know its value $S_e(T)$ at T, which we obtain from the intratemporal condition.

Let us denote by Z the per-unit penalty for excess emissions over the limit T. We assume that Z is constant. Let us also denote by $S_1(t)$ (resp. $S_2(t)$) the price of fuel 1 (resp. 2) at date t. Then, if Z is sufficiently large so that it is not binding, the intratemporal condition, or the equality of marginal costs for fuel and emission allowances, leads to the equality of the allowance price and a spread between fuel prices at T.[4] Denote this spread by $H_1S_1(T) - H_2S_2(T)$, where we assume H_1 and H_2 are constant for simplicity. On the other hand, if the spread is larger than the penalty Z, i.e., $H_1S_1(T) - H_2S_2(T) > Z$, the emission allowance price cannot be equal to the spread; otherwise, some financial institutions will short sell emission allowances and pay the penalty Z that is less than the spread or the emission allowance price. Hence, the penalty Z sets the upper bound of the emission allowance price at T. Furthermore, the emission allowance price cannot be negative. Thus, the emission allowance price at T is given by

$$S_e(T) = [\{H_1S_1(T) - H_2S_2(T)\} \wedge Z] \vee 0 \tag{2}$$

where $a \wedge b = \min\{a, b\}$ and $a \vee b = \max\{a, b\}$.

Finally, to describe the spot commodity prices, we assume that commodity prices follow the Gibson–Schwartz (1990) model. That is, for a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$, we assume that the fuel prices $S_i(t)$ and convenience yields $\delta_i(t)$ satisfy the following stochastic differential equations.

$$dS_i(t) = S_i(t)(r - \delta_i(t))dt + S_i(t)\sigma_{S_i}dW_{S_i}(t) \tag{3}$$
$$d\delta_i(t) = \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i}dW_{\delta_i}(t) \tag{4}$$

where W are four-dimensional standard Brownian motions under the risk-neutral probability.[5]

To summarize, we assume three conditions. The first is the intertemporal condition that emission allowances at time t are the present value of the emission allowances at the end of period T. The second is the intratemporal condition that the emission allowance spot price at the end of period T should

[4]This can be understood intuitively as follows. Let h_1 (resp. h_2) be the amount of fuel 1 (resp. 2) necessary for producing one marginal unit of electricity. Let k_1 (resp. k_2) be the amount of CO_2 emissions associated with burning one marginal unit of fuel 1 (resp. 2). Then, the equality of marginal costs implies $h_1S_1(T) + h_1k_1S_e(T) = h_2S_2(T) + h_2k_2S_e(T)$, which leads to $S_e(T) = \frac{1}{h_2k_2 - h_1k_1}(h_1S_1(T) - h_2S_2(T))$. See Nakajima and Ohashi (2010) for more general cases.

[5]The volatility structure is given in the Appendix.

be positive and equal to the minimum of the spread of the two fuel prices and the penalty. The last is that commodity prices follow the Gibson–Schwartz model.

2.2 Spot Price of Emission Allowances

Under the assumptions above, we can derive the spot price of emission allowances as follows.

Proposition 2.1.

Under assumptions (1)–(4), the spot price of emission allowances is given by

$$S_e(t) \quad = \quad \hat{H}_1(t,T)S_1(t) - \hat{H}_2(t,T)S_2(t) + \hat{H}_3(t,T)Z \tag{5}$$

where $\hat{H}_i(t,T)$ are defined in the Appendix.

Proof. The proof is in the Appendix. □

Thus, if fuel switching by producers is the main factor of the emission allowance price, and if we can regard other factors as negligible, the emission allowance price is expressed as a spread of fuel prices with the penalty appropriately discounted.

Observe, however, that the spread relation is not simple: even if the coefficients (H_1 and H_2) on fuel prices in the spread are constant at the end of trading period T, the corresponding coefficients ($\hat{H}_1(t,T)$ and $\hat{H}_2(t,T)$) in the spread that determines the emission allowance price in (5) are not constant and depend on the stochastic properties of fuel prices as well as time to maturity. Indeed, $\hat{H}_1(t,T)$ (resp. $\hat{H}_2(t,T)$) can be interpreted as H_1 (resp. H_2) multiplied by the discount factor and the risk-adjusted probability of the spread $H_1S_1(T) - H_2S_2(T)$ of fuel prices between 0 and Z at date T.

One implication of this expression is that the emission allowance spot price may inherit stochastic properties from the fuel prices. For example, while the coefficients ($\hat{H}_1(t,T)$ and $\hat{H}_2(t,T)$) are changing stochastically over time, if the fuel prices exhibit convenience yields, the emission allowance price may also exhibit a convenience yield through the spread relation in (5).

The emission allowance price also depends on the penalty Z multiplied by $\hat{H}_3(t,T)$ because the producers have the option of emitting any amount of CO_2 by paying the penalty Z per unit of emission at date T. Here, the coefficient $\hat{H}_3(t,T)$ can be interpreted as the adjusted discount factor that consists of the risk-free discount rate multiplied by the risk-neutral probability that the spread $H_1S_1(T) - H_2S_2(T)$ of fuel prices exceeds Z at date T.

This result implies that the value of emission allowances can be replicated by holding $\hat{H}_1(t,T)$ units of fuel 1, short-selling $\hat{H}_2(t,T)$ units of fuel 2, and holding $\hat{H}_3(t,T)Z$ units in risk-free assets at date t. Hence, a power company can control its exposure to emission allowance price risk by using this formula to adjust its position on fuels. We discuss the hedging strategy in the next section.

2.3 The Option Value Embedded in the Emission Allowance Spot Price

From equations (1) and (2), the value of the emission allowance can be replicated by a portfolio of buying 1 unit of European call options with exercise price 0 on the spread $H_1S_1(T) - H_2S_2(T)$ at the maturity date T and selling 1 unit of European calls with exercise price Z on the spread. That is, the emission allowance is the bull call spread of the options on the spread $H_1S_1(T) - H_2S_2(T)$ at date T.

In this section, we analyze the values of these embedded options in more detail. For this purpose, we first derive the valuation formula for the emission allowance spot price when the option to emit CO_2 by paying penalty Z is ignored, or when Z is taken to be infinite, as follows.

Proposition 2.2. Define

$$S'_e(t) \equiv E_t[e^{-r(T-t)}((H_1S_1(T) - H_2S_2(T)) \vee 0)].$$

Then

$$S'_e(t) = H'_1(t,T)S_1(t) - H'_2(t,T)S_2(t) \tag{6}$$

where

$$H'_1(t,T) = H_1 \exp\left\{ -r(T-t) + \mu_{X_1}(t,T) + \frac{1}{2}\sigma^2_{X_1}(t,T) \right\}(1 - \Phi(\hat{\mu}_1(t,T)))$$

$$H'_2(t,T) = H_2 \exp\left\{ -r(T-t) + \mu_{X_2}(t,T) + \frac{1}{2}\sigma^2_{X_2}(t,T) \right\}(1 - \Phi(\hat{\mu}_2(t,T)))$$

where $\Phi(\cdot)$ is the standard normal distribution function.

Proof. The derivation is similar to that of Proposition 2.1 and hence we omit.[6] □

[6] The proof is provided upon request.

Notice that equation (6) can be regarded as the present value of the maximum of the spread of two commodity prices and 0 at the end of period T. Thus, the embedded option value for penalty Z of the emission allowance spot price is expressed by the difference between $S'_e(t)$ and $S_e(t)$.

Corollary 2.1. The option value of penalty Z embedded in the emission allowance is given by

$$\begin{aligned} S_e(t) - S'_e(t) &= \exp\left\{-r(T-t) + \mu_{X_1}(t,T) + \frac{1}{2}\sigma^2_{X_1}(t,T)\right\} H_1 S_1(t) E_1 \\ &\quad - \exp\left\{-r(T-t) + \mu_{X_2}(t,T) + \frac{1}{2}\sigma^2_{X_2}(t,T)\right\} H_2 S_2(t) E_2 \\ &\quad + \hat{H}_3(t,T) Z \end{aligned}$$

where

$$\begin{aligned} E_1 &= -\int_{-\infty}^{\infty} \Phi(-d_1(x_2, Z)) n(x_2 | \mu_{X_2}(t,T) + \sigma_{X_1 X_2}(t,T), \sigma^2_{X_2}(t,T)) dx_2 \\ E_2 &= -\int_{-\infty}^{\infty} \Phi(-d_2(x_2, Z)) n(x_2 | \mu_{X_2}(t,T) + \sigma^2_{X_2}(t,T), \sigma^2_{X_2}(t,T)) dx_2. \end{aligned}$$

and $n(\cdot|\mu, \sigma^2)$ is the normal density function with μ and σ^2 as mean and variance, respectively.

This is the value of the option to emit any amount of CO_2 by paying penalty Z when the spread $H_1 S_1(T) - H_2 S_2(T)$ exceeds Z. From this corollary, we can see that the option value for penalty Z embedded in the emission allowance spot price is affected not only by penalty Z, but also by $S_1(t)$ and $S_2(t)$ through E_i $(= 1, 2)$.

Similarly, if the investor (mis-)values the emission allowance spot price just as the spread of two commodity prices, he/she will be subject to the next equation.

Proposition 2.3. Define

$$S''_e(t) \equiv E_t[e^{-r(T-t)}(H_1 S_1(T) - H_2 S_2(T))].$$

Then

$$\begin{aligned} S''_e(t) &= \exp\left\{ - r(T-t) + \mu_{X_1}(t,T) + \frac{1}{2}\sigma^2_{X_1}(t,T)\right\} H_1 S_1(t) \\ &\quad - \exp\left\{ - r(T-t) + \mu_{X_2}(t,T) + \frac{1}{2}\sigma^2_{X_2}(t,T)\right\} H_2 S_2(t). \end{aligned}$$

Proof. The formula for $S''_e(t)$ is obvious by the linearity of expectations. □

Equation (7) is merely the present value of the spread of two commodity prices. Comparing equation (7) with $S'_e(t)$, we have the following result.

Corollary 2.2.

$$\begin{aligned} & S'_e(t) - S''_e(t) \\ = \; & -\exp\left\{ -r(T-t) + \mu_{X_1}(t,T) + \frac{1}{2}\sigma^2_{X_1}(t,T) \right\} H_1 S_1(t) \Phi(\hat{\mu}_1(t,T)) \\ & + \exp\left\{ -r(T-t) + \mu_{X_2}(t,T) + \frac{1}{2}\sigma^2_{X_2}(t,T) \right\} H_2 S_2(t) \Phi(\hat{\mu}_2(t,T)) \end{aligned}$$

Thus, the option value of emission allowances against the spread of the two commodity prices can be decomposed into two components. The first component $S_e(t) - S'_e(t)$ is the option value of the emission allowance for penalty Z and the second component $S'_e(t) - S''_e(t)$ is the option value of the emission allowance for exercise price 0.[7]

2.4 Derivatives of Emission Allowances

Given the spot price, we can derive the prices of emission allowance derivatives. First, we calculate the emission allowance futures price in the following proposition.

Proposition 2.4. The futures price for an emission allowance that matures at T is

$$\begin{aligned} G_e(t,T) &= E_t[S_e(T)] \\ &= E_t[((H_1 S_1(T) - H_2 S_2(T)) \wedge Z) \vee 0] \\ &= e^{r(T-t)} \hat{H}_1(t,T) S_1(t) - e^{r(T-t)} \hat{H}_2(t,T) S_2(t) + e^{r(T-t)} \hat{H}_3(t,T) Z. \end{aligned}$$

Proof. The first equation is from Cox, Ingersoll, and Ross (1981). The proof for the third equation is the same as Proposition 2.1. □

Next, we obtain the valuation formula for a European call option of emission allowances.

[7] From another point of view, $S_e(t) - S'_e(t)$ and $S_e(t) - S''_e(t) = S_e(t) - S'_e(t) + S'_e(t) - S''_e(t)$ can be interpreted as pricing errors for the emission allowance price when $((H_1 S_1(T) - H_2 S_2(T)) \wedge Z) \vee 0$ is replaced by $(H_1 S_1(T) - H_2 S_2(T)) \vee 0$ or $H_1 S_1(T) - H_2 S_2(T)$, i.e., when the investor misprices the emission allowance spot price at the end of period T.

Proposition 2.5. Suppose $Z > K > 0$. The option price for the emission allowance that matures at T is

$$\begin{aligned} C_e(t,T) &= E_t[e^{-r(T-t)}(S_e(T)-K)^+] \\ &= E_t[e^{-r(T-t)}(((H_1S_1(T)-H_2S_2(T)-K)\wedge(Z-K))\vee 0)] \\ &= \bar{H}_1(t,T)S_1(t)-\bar{H}_2(t,T)S_2(t)-\bar{H}_3(t,T)K+\bar{H}_4(t,T)Z \end{aligned}$$

where

$$\begin{aligned} \bar{H}_1(t,T) &= H_1\exp\left\{-r(T-t)+\mu_{X_1}(t,T)+\frac{1}{2}\sigma^2_{X_1}(t,T)\right\} \\ &\quad\times\int_{-\infty}^{\infty}(\Phi(d_1(x_2,Z))-\Phi(d_1(x_2,K))) \\ &\quad\times n(x_2|\mu_{X_2}(t,T)+\sigma_{X_1X_2}(t,T),\sigma^2_{X_2}(t,T))dx_2 \end{aligned}$$

$$\begin{aligned} \bar{H}_2(t,T) &= H_2\exp\left\{-r(T-t)+\mu_{X_2}(t,T)+\frac{1}{2}\sigma^2_{X_2}(t,T)\right\} \\ &\quad\times\int_{-\infty}^{\infty}(\Phi(d_2(x_2,Z))-\Phi(d_2(x_2,K))) \\ &\quad\times n(x_2|\mu_{X_2}(t,T)+\sigma^2_{X_2}(t,T),\sigma^2_{X_2}(t,T))dx_2 \end{aligned}$$

$$\begin{aligned} \bar{H}_3(t,T) &= \exp(-r(T-t))\int_{-\infty}^{\infty}(\Phi(d_2(x_2,Z)-\Phi(d_2(x_2,K))) \\ &\quad\times n(x_2|\mu_{X_2}(t,T),\sigma^2_{X_2}(t,T))dx_2 \\ \bar{H}_4(t,T) &= \exp(-r(T-t))\int_{-\infty}^{\infty}(1-\Phi(d_2(x_2,Z)))n(x_2|\mu_{X_2}(t,T),\sigma^2_{X_2}(t,T))dx_2 \end{aligned}$$

Proof. Again, the derivation is similar to that of Proposition 2.1 and hence we omit.[8] □

Trivially, if $Z \leq K$ then the option price is 0, and if $K \leq 0$ then the option price is the same as the emission allowance spot price.

As the penalty is paid at the end of period T, a firm such as a power company that needs to hedge the penalty will naturally focus on the payment at time T. Therefore, derivatives of the emission allowances that mature at T should be adequate for risk hedging. Note also that these derivatives are derivatives on a derivative on the spread of two commodities.

[8] The proof is provided upon request.

3 Hedging Emission Allowances Using Commodity Futures

Equation (5) shows that the value of emission allowances at date t can be replicated by holding $\hat{H}_1(t,T)$ units of fuel 1, short-selling $\hat{H}_2(t,T)$ units of fuel 2, and holding $\hat{H}_3(t,T)Z$ units in risk-free assets at date t. It turns out, however, that obtaining a hedging strategy to replicate emission allowances from this relation is a difficult task because of the dependency of coefficients $\hat{H}_i(t,T)$ $(i=1,2,3)$ on the commodity prices and the time to maturity.

On the other hand, because the compliance of emission reductions is checked and the penalty is paid only at the end of period T, a firm that needs to hedge the penalty only has to care the payment to emission allowances at time T. Thus, the derivatives of emission allowances that mature at T should be enough for its risk hedging. Moreover, the commodity futures can be traded more easily than their spots. Hence, in this section, we investigate the hedging strategy to replicate the emission allowance futures with maturity T by using the commodity futures with the same maturity.

We can derive the hedging strategy for the emission allowance futures as follows.

Proposition 3.1. Assume (1)–(4). The hedging equation for emission allowances using commodity futures is

$$dG_e(t,T) \;=\; \varphi_B(t)dB(t) + \varphi_{G_1}(t)dG_1(t,T) + \varphi_{G_2}(t)dG_2(t,T)$$

where the hedging strategies are

$$\begin{aligned}
\varphi_B(t) \;=\; & \Bigg\{ \frac{\partial \hat{\hat{H}}_1(t,T)}{\partial t} G_1(t,T) - \frac{\partial \hat{\hat{H}}_2(t,T)}{\partial t} G_2(t,T) + Z \frac{\partial \hat{\hat{H}}_3(t,T)}{\partial t} \\
& + \sum_{j,k=1}^{2} \Bigg(\frac{G_1(t,T)}{2} \frac{\partial^2 \hat{\hat{H}}_1(t,T)}{\partial G_j(t,T)\partial G_k(t,T)} - \frac{G_2(t,T)}{2} \frac{\partial^2 \hat{\hat{H}}_2(t,T)}{\partial G_j(t,T)\partial G_k(t,T)} \\
& + \frac{Z}{2} \frac{\partial^2 \hat{\hat{H}}_3(t,T)}{\partial G_j(t,T)\partial G_k(t,T)} \Bigg) \Bigg(\sigma_{S_j S_k} G_j(t,T) G_k(t,T) \\
& - \sigma_{S_j \delta_k} \frac{1-e^{-\kappa_k (T-t)}}{\kappa_k} G_j(t,T) G_k(t,T) - \sigma_{S_k \delta_j} \frac{1-e^{-\kappa_j (T-t)}}{\kappa_j} G_j(t,T) G_k(t,T) \\
& + \sigma_{\delta_j \delta_k} \frac{(1-e^{-\kappa_j (T-t)})(1-e^{-\kappa_k (T-t)})}{\kappa_j \kappa_k} G_j(t,T) G_k(t,T) \Bigg)
\end{aligned}$$

$$
\begin{aligned}
&+\frac{\partial\hat{\hat{H}}_1(t,T)}{\partial G_1(t,T)}\Bigg(\sigma_{S_1}^2 G_1^2(t,T) - 2\sigma_{S_1\delta_1}\frac{1-e^{-\kappa_1(T-t)}}{\kappa_1}G_1^2(t,T)\\
&+\sigma_{\delta_1}^2\frac{(1-e^{-\kappa_1(T-t)})^2}{\kappa_1^2}G_1^2(t,T)\Bigg)\\
&+\frac{\partial\hat{\hat{H}}_1(t,T)}{\partial G_2(t,T)}\Bigg(\sigma_{S_1S_2}G_1(t,T)G_2(t,T) - \sigma_{S_1\delta_2}\frac{1-e^{-\kappa_2(T-t)}}{\kappa_2}G_1(t,T)G_2(t,T)\\
&-\sigma_{S_2\delta_1}\frac{1-e^{-\kappa_1(T-t)}}{\kappa_1}G_1(t,T)G_2(t,T)\\
&+\sigma_{\delta_1\delta_2}\frac{(1-e^{-\kappa_1(T-t)})(1-e^{-\kappa_2(T-t)})}{\kappa_1\kappa_2}G_1(t,T)G_2(t,T)\Bigg)\\
&-\frac{\partial\hat{\hat{H}}_2(t,T)}{\partial G_2(t,T)}\Bigg(\sigma_{S_2}^2 G_2^2(t,T) - 2\sigma_{S_2\delta_2}\frac{1-e^{-\kappa_2(T-t)}}{\kappa_2}G_2^2(t,T)\\
&+\sigma_{\delta_2}^2\frac{(1-e^{-\kappa_2(T-t)})^2}{\kappa_2^2}G_2^2(t,T)\Bigg)\\
&-\frac{\partial\hat{\hat{H}}_2(t,T)}{\partial G_1(t,T)}\Bigg(\sigma_{S_1S_2}G_1(t,T)G_2(t,T) - \sigma_{S_1\delta_2}\frac{1-e^{-\kappa_2(T-t)}}{\kappa_2}G_1(t,T)G_2(t,T)\\
&-\sigma_{S_2\delta_1}\frac{1-e^{-\kappa_1(T-t)}}{\kappa_1}G_1(t,T)G_2(t,T)\\
&+\sigma_{\delta_1\delta_2}\frac{(1-e^{-\kappa_1(T-t)})(1-e^{-\kappa_2(T-t)})}{\kappa_1\kappa_2}G_1(t,T)G_2(t,T)\Bigg)\Bigg\}(rB(t))^{-1}dt
\end{aligned}
$$

$$
\begin{aligned}
\varphi_{G_i}(t) &= \sum_{j=1}^{2}\frac{\partial\hat{\hat{H}}_j(t,T)}{\partial G_i(t,T)}G_j(t,T)dG_i(t,T) + \hat{\hat{H}}_i(t,T)dG_i(t,T)\\
&-\sum_{j=1}^{2}\frac{\partial\hat{\hat{H}}_j(t,T)}{\partial G_i(t,T)}G_j(t,T)dG_i(t,T) + \frac{\partial\hat{\hat{H}}_3(t,T)}{\partial G_i(t,T)}ZdG_i(t,T), i=1,2
\end{aligned}
$$

Proof. The proof is in the Appendix. □

Note that because the spot price of emission allowances is equal to their futures price at maturity, an investor, say a power company, that needs to hedge the allowance at maturity can satisfy its need by hedging its futures. The advantage of hedging the futures is that we need to use only two futures of the commodities with the same maturity as the emission allowance futures. With the same maturity, there is no convenience yield and we do not need to control it using other commodity futures. This greatly simplifies the calculation.

4 Numerical Analysis

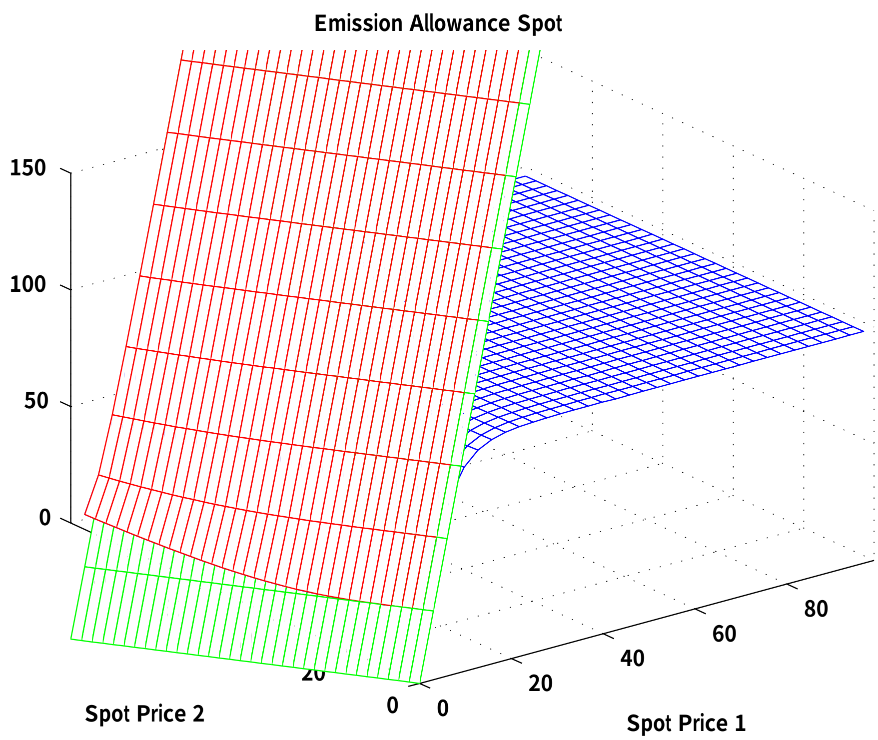
We conduct a numerical analysis assuming the following parameter values. The spot prices of natural gas and coal are 10 and 70 euros, respectively, with no convenience yield at time t. The heat/emission ratios H_1 and H_2 are 10 and 0.50, respectively. The penalty is 100 euros and the period is set to one year. The volatilities of the spot prices of natural gas and coal are 0.40 and 0.50 and the volatilities of the convenience yields are 0.40 and 0.30, respectively. The other parameters are summarized as follows.

$$\begin{aligned}
&S_1(t) = 10, S_2(t) = 70,\\
&\delta_1(t) = 0.00, \delta_2(t) = 0.00,\\
&\sigma_{S_1} = 0.40, \sigma_{S_2} = 0.50, \sigma_{\delta_1} = 0.40, \sigma_{\delta_2} = 0.30,\\
&\rho_{S_1S_2} = 0.90, \rho_{S_1\delta_1} = 0.10, \rho_{S_1\delta_2} = 0.00,\\
&\rho_{S_2\delta_1} = -0.20, \rho_{S_2\delta_2} = 0.10,\\
&\rho_{\delta_1\delta_2} = 0.00,\\
&\kappa_1 = 2.00, \kappa_2 = 1.00,\\
&\hat{\alpha}_1 = 0.10, \hat{\alpha}_2 = 0.30,\\
&T = 1, Z = 100, r = 0.04,\\
&H_1 = 10, H_2 = 0.50.
\end{aligned}$$

Figure 1 illustrates theoretical emission allowance spot prices $S_e(t)$, $S'_e(t)$, and $S''_e(t)$. Commodity spot price $S_1(t)$ affects the emission allowance spot price more than commodity spot price $S_2(t)$ does. This is because H_1 is larger than H_2. Comparing $S_e(t)$ with $S'_e(t)$ and $S''_e(t)$, $S_e(t)$ is consistently lower than $S'_e(t)$ and $S''_e(t)$ because the emission allowance spot prices have the upper limit Z. Furthermore, the differences between $S_e(t)$ and $S'_e(t)$ or $S''_e(t)$ are relatively large compared with that between $S'_e(t)$ and $S''_e(t)$. This suggests that ignoring the option value of the penalty leads to larger errors than ignoring the nonnegativity of the emission allowance spot price.

The sensitivity of the emission allowance spot price to σ_{S_1} and σ_{S_2} is shown in Figure 2. Because the emission allowance is a spread option, its sensitivity to volatility of spot prices differs from that of a plain vanilla option. In our case, as σ_{S_1} increases, the emission allowance spot price decreases, and σ_{S_2} has the opposite effect. These effects vary when the volatility structure changes. Furthermore, note that as σ_{S_2} increases, the value of embedded options decreases. However, as σ_{S_1} increases, the value of embedded options

Figure 1: Sensitivity of the emission allowance spot price to commodity spot prices. The lowest, highest, and middle surfaces represent the theoretical emission allowance spot prices $S_e(t)$, $S'_e(t)$, and $S''_e(t)$, respectively.



increases. and this increase in value is much larger than the decrease in value for σ_{S_2}. This is because $H_1S_1(t)$ is larger than $H_2S_2(t)$ and there is more room for the value $H_1S_1(t)$ to fluctuate.

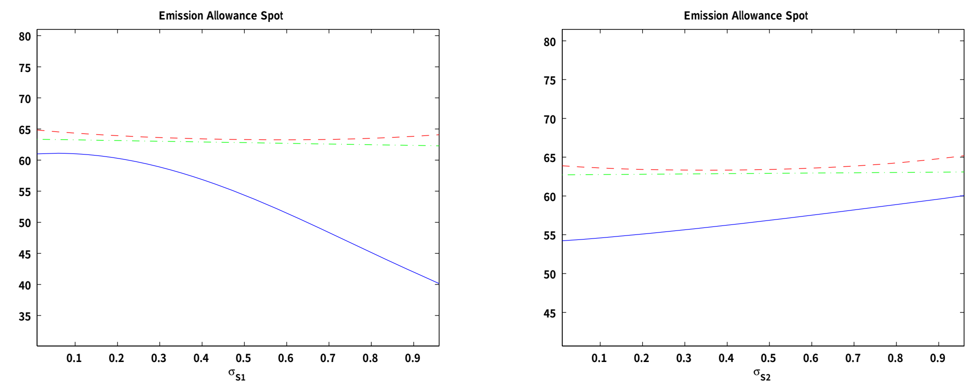


Figure 2: Sensitivity of the emission allowance spot price to σ_{S_1} and σ_{S_2}. The solid line, the dashed line, and the chain line represent the theoretical emission allowance spot prices $S_e(t)$, $S'_e(t)$, and $S''_e(t)$, respectively.

Figures 3 plot the price of the commodity spread option with σ_{δ_1} and σ_{δ_2}. For σ_{δ_1}, the emission allowance spot price $S_e(t)$ is an inverted U-shaped. Meanwhile, as σ_{δ_2} increases, the emission allowance spot price decreases.

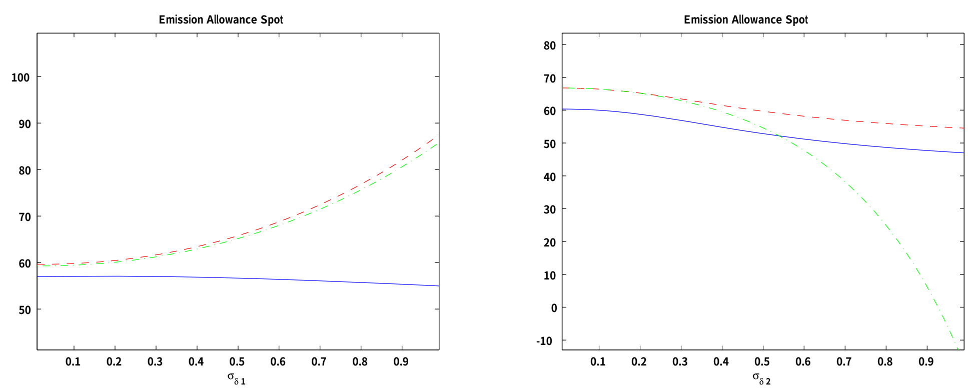


Figure 3: Sensitivity of the emission allowance spot price to σ_{δ_1} and σ_{δ_2}. The solid line, the dashed line, and the chain line represent the theoretical emission allowance spot prices $S_e(t)$, $S'_e(t)$, and $S''_e(t)$, respectively.

The results for the long-term mean of the convenience yield $\hat{\alpha}_i$ are presented in Figure 4. As the long-term mean $\hat{\alpha}_1$ of the convenience yield for commodity 1 increases, the emission allowance spot price decreases. This is true for $S'_e(t)$ and $S''_e(t)$. Observe that the emission allowance spot price converges to 0 when $\hat{\alpha}_1$ increases. This can be explained in terms of the dividend rate. If the long-term mean of the dividend rate increases, the price will fall more steeply after the ex-dividend date. Because the convenience yield can be regarded as the dividend rate, the same reason applies. On the other hand, the emission allowance spot price increases as $\hat{\alpha}_2$ increases. This is because $H_2 S_2(t)$ affects the emission allowance price negatively.

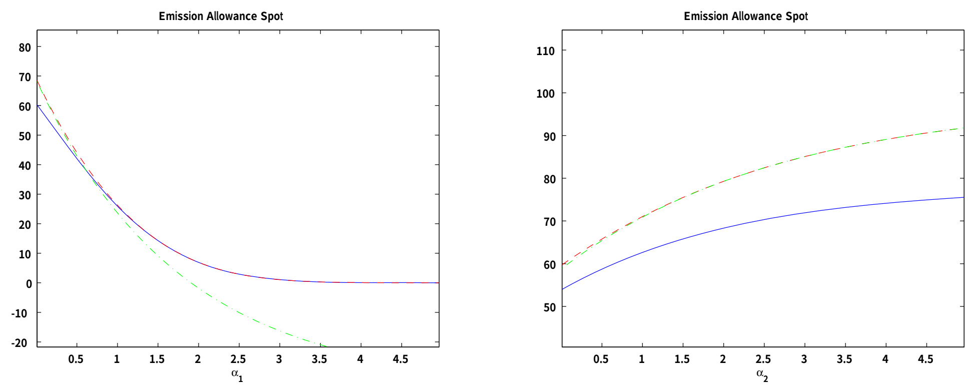


Figure 4: Sensitivity of the emission allowance spot price to a_1 and a_2. The solid line, the dashed line, and the chain line represent the theoretical emission allowance spot prices $S_e(t)$, $S'_e(t)$, and $S''_e(t)$, respectively.

Figure 5 shows the sensitivity of the emission allowance spot price to κ_i. Both κ_i have the same effect on the emission allowance spot price. However, for $S'_e(t)$ and $S''_e(t)$, the sensitivity to κ_1 has opposite effects to those for $S_e(t)$.

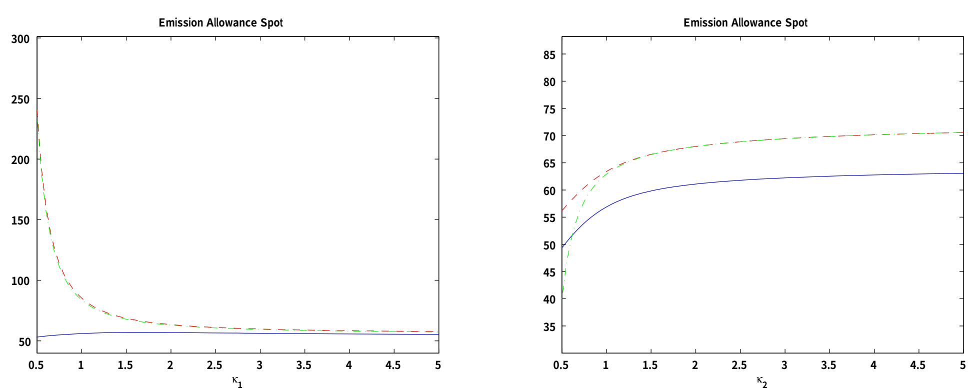
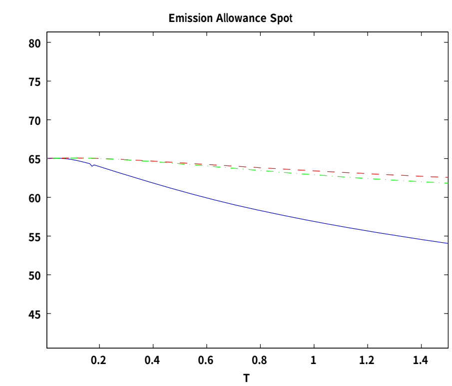


Figure 5: Sensitivity of the emission allowance spot price to κ_1 and κ_2. The solid line, the dashed line, and the chain line represent theoretical emission allowance spot prices $S_e(t)$, $S'_e(t)$, and $S''_e(t)$, respectively.

Figures 6, 7, and 8 show the sensitivity to maturity, the interest rate, and the penalty, respectively. The emission allowance spot prices $S_e(t)$, $S'_e(t)$, and $S''_e(t)$ decrease as maturity increases, which is of course due to the discount effect of the interest rate. Moreover, as the interest rate increases, the emission allowance spot prices $S_e(t)$, $S'_e(t)$, and $S''_e(t)$ decrease. The result on the penalty Z is also obvious. As Z increases, the emission allowance spot price $S_e(t)$ also increases. However, $S'_e(t)$ and $S''_e(t)$ are not affected by the penalty.

Figure 6: Sensitivity of the emission allowance spot price to maturity. The solid line, the dashed line, and the chain line represent the theoretical emission allowance spot prices $S_e(t)$, $S'_e(t)$, and $S''_e(t)$, respectively.



We now turn to the analysis of hedge ratios. Figure 9 shows the sensitivity of hedge ratios of commodity futures to commodity futures prices. Comparing the hedge ratios, we see that the hedge ratio of G_1 (the upper surface) is more sensitive than that of G_2 (the lower surface). This is because H_1 is larger than H_2, which implies that $G_1(t,T)$ has more impact on the hedge ratio than $G_2(t,T)$ does. Furthermore, the hedge ratio of G_1 is more sensitive to $G_1(t,T)$ than to $G_2(t,T)$. Note that as $G_1(t,T)$ gets large, the emission allowance futures price converges to $e^{r(T-t)}Z$ and thus the hedge ratios converge to zero.

Figure 10 shows the sensitivity of the hedge ratios to the volatilities of commodity prices. The hedge ratios of G_1 do not increase or decrease monotonically as σ_{S_i} $(i = 1, 2)$ increases. σ_{S_i} affects the hedge ratios through $\mu_{\hat{X}_{G_i}}$ and $\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}}$ with correlations ρ_{S_i,S_j} and ρ_{S_i,δ_i}. Therefore, it is difficult to determine the direction of changes in the hedge ratios as the volatility changes. The effect can be negative or positive. Likewise, the hedge ratios of G_2 increase and then decrease as σ_{S_i} increases. However, because H_1 is larger than H_2, the hedge ratio of G_2 is insensitive to both parameters.

Figure 11 shows the sensitivity of the hedge ratios to σ_{δ_1} and σ_{δ_2}. We see that the hedge ratio of G_1 has an U-shape relative to σ_{δ_1} and σ_{δ_2}. σ_{δ_i} has the

Figure 7: Sensitivity of the emission allowance spot price to interest rate. The solid line, the dashed line, and the chain line represent the theoretical emission allowance spot prices $S_e(t)$, $S'_e(t)$, and $S''_e(t)$, respectively.

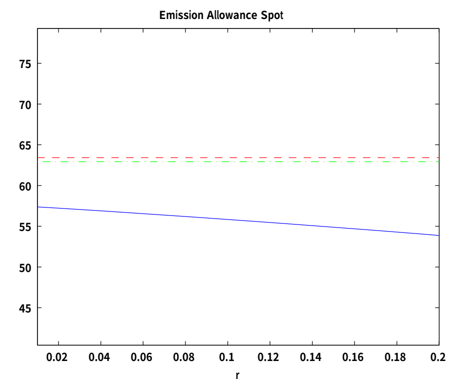


Figure 8: Sensitivity of the emission allowance spot price to penalty. The solid line, the dashed line, and the chain line represent the theoretical emission allowance spot prices $S_e(t)$, $S'_e(t)$, and $S''_e(t)$, respectively.

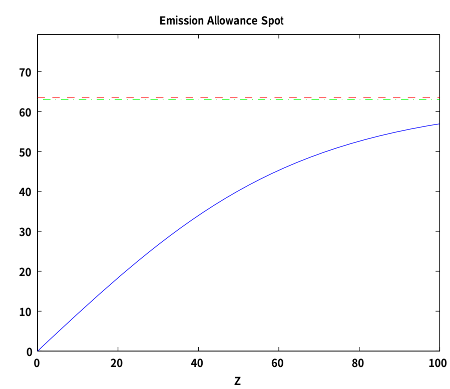
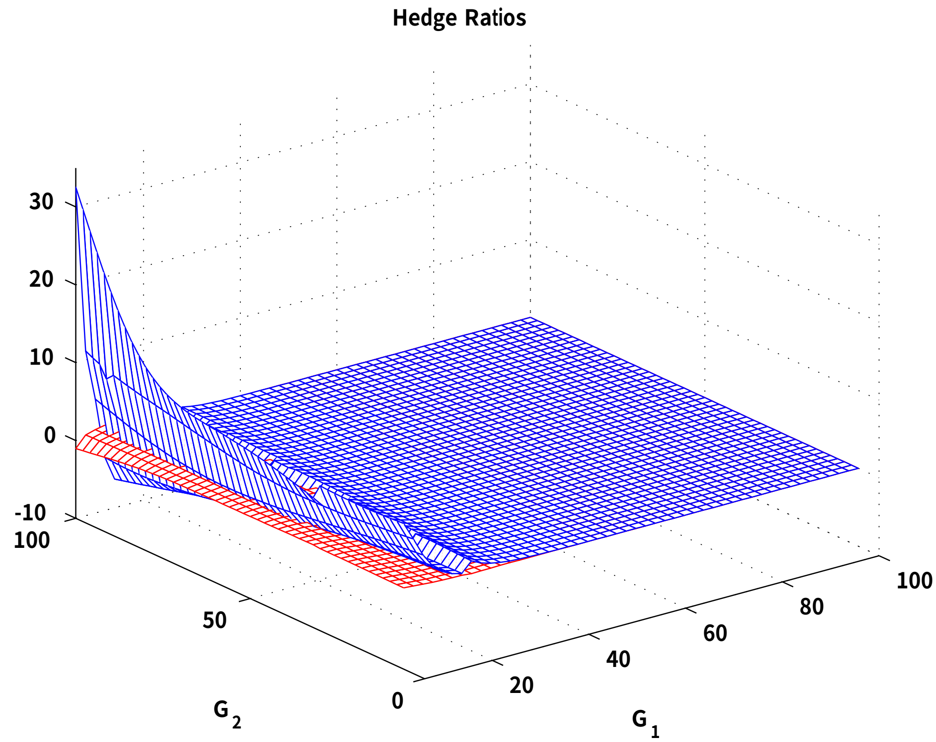


Figure 9: Sensitivity of hedge ratios of commodity futures to $G_1(t,T)$ and $G_2(t,T)$. The upper and lower surfaces represent the hedge ratio of commodity futures 1 and the hedge ratio of commodity futures 2, respectively.



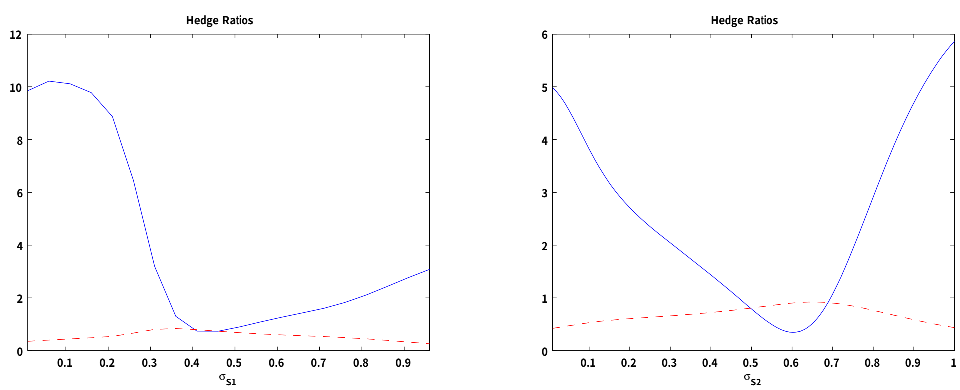


Figure 10: Sensitivity of hedge ratios of commodity futures to σ_{S_1} and σ_{S_2}. The solid line and the dashed line represent the hedge ratio of commodity futures 1 and the hedge ratio of commodity futures 2, respectively.

same effect as σ_{S_i} on $\mu_{\hat{X}_{G_i}}$ and $\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}}$, which causes the complicated relation. Again, the hedge ratio of G_2 is insensitive to the volatility parameters, for the reason we have just described.

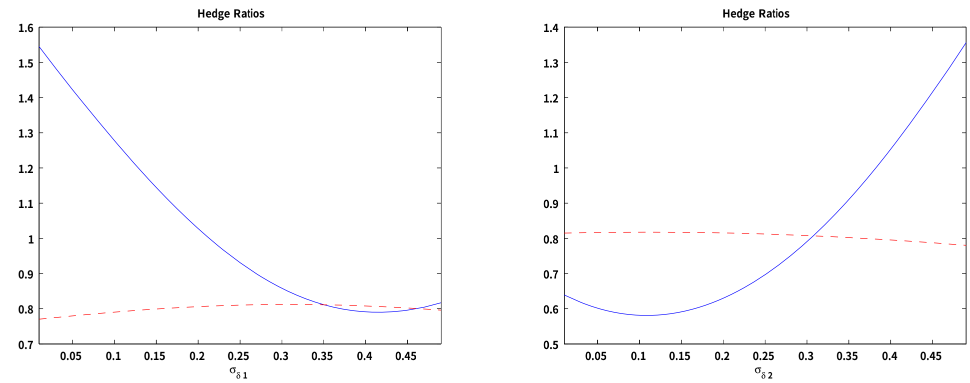


Figure 11: Sensitivity of hedge ratios of commodity futures to σ_{δ_1} and σ_{δ_2}. The solid line and the dashed line represent the hedge ratio of commodity futures 1 and the hedge ratio of commodity futures 2, respectively.

Figure 12 shows the sensitivity of the hedge ratios to κ_i $(i = 1, 2)$. We see that there are same effect on the hedge ratio of G_1 for κ_1 as σ_{δ_1}, which can be explained by the same argument as above. The hedge ratio of G_1 decreases as κ_2 increases. On the other hand, because H_1 is larger than H_2, the hedge ratio of G_2 is smaller than that of G_1.

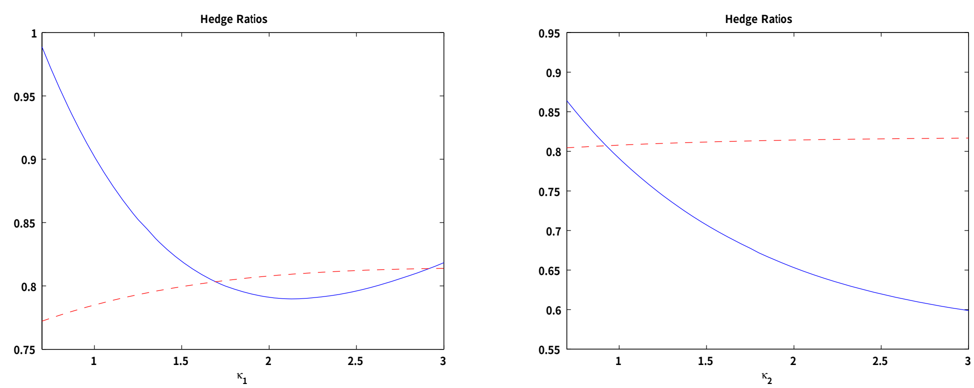


Figure 12: Sensitivity of hedge ratios of commodity futures to κ_1 and κ_2. The solid line and the dashed line represent the hedge ratio of commodity futures 1 and the hedge ratio of commodity futures 2, respectively.

5 Conclusion

In this paper, we proposed a model of the emission allowance spot price as a derivative of the commodity spread. We assumed that the emission allowance spot price at the end of the trading period was equal to the minimum of the spread of the two commodity prices and the penalty when it was positive, or equal to zero otherwise. We emphasized the interrelation between prices of emission allowances and commodities (i.e., fuels), which had not been incorporated in preceding papers on the valuation of emission allowances.

This paper showed that the emission allowance spot price could be replicated as the value of a portfolio of commodities and a risk-free asset. This formula provides a firm, such as a power company, with a hedging strategy for the required emission allowance. Furthermore, the formula implies that the emission allowance spot price could exhibit properties similar to those of commodity prices, because the emission allowance price can be replicated as a portfolio of two commodities. We characterized the values of options embedded in emission allowances. In addition, we derived the formulae for emission allowance futures and options. We also analyzed the behavior of the hedge ratios of emission allowance futures by commodity futures. From the numerical analysis with certain parameter values, we found that the option values for the penalty embedded in emission allowances was relatively large, which implied that the penalty was an important component in evaluating emission allowances.

For future research, it would be interesting to conduct an empirical analysis of the valuation formula derived in this paper. Heat/emission ratios can be included in the critical parameters in such an analysis. It would also be interesting to explore the model using alternative assumptions. For example, we could investigate a model in which the emission allowance price at the end of the period is determined in a different way to that assumed in this paper. We could also analyze a model whose underlying commodity prices follow stochastic processes that are different from the standard Gibson–Schwartz process and may include seasonality, jumps, or stochastic volatility. As we emphasized, the interrelation between the prices of emission allowances and commodities (e.g., fuels) should be the key to understanding the properties of emission allowance prices. With this point in mind, empirical analyses on the prices of coal, natural gas, electricity and other commodities should form the foundation for the study of emission allowance prices.

6 Appendices

6.1 The Solutions of Spot Prices

In this subsection, we provide the solutions of spot prices and its statistical properties. The spot prices can be derived as follows.[9]

$$
\begin{aligned}
S_i(T) &= S_i(t)\exp\left\{\left(r-\frac{1}{2}\sigma_{S_i}^2-\hat{\alpha}_i\right)(T-t)+\frac{1}{\kappa_i}(\hat{\alpha}_i-\delta_i(t))(1-e^{-\kappa_i(T-t)})\right.\\
&\quad\left.+\int_t^T\sigma_{S_i}dW_{S_i}(s)-\frac{1}{\kappa_i}\int_t^T\sigma_{\delta_i}dW_{\delta_i}(s)+\frac{1}{\kappa_i}\int_t^T e^{-\kappa_i(T-s)}\sigma_{\delta_i}dW_{\delta_i}(s)\right\}\\
&= S_i(t)\exp\{X_i(t,T)\}
\end{aligned}
$$

where

$$
\begin{aligned}
X_i(t,T) &\triangleq \left(r-\frac{1}{2}\sigma_{S_i}^2-\hat{\alpha}_i\right)(T-t)+\frac{1}{\kappa_i}(\hat{\alpha}_i-\delta_i(t))(1-e^{-\kappa_i(T-t)})\\
&\quad+\int_t^T\sigma_{S_i}dW_{S_i}(s)-\frac{1}{\kappa_i}\int_t^T\sigma_{\delta_i}dW_{\delta_i}(s)+\frac{1}{\kappa_i}\int_t^T e^{-\kappa_i(T-s)}\sigma_{\delta_i}dW_{\delta_i}(s)
\end{aligned}
$$

Notice that $X_i(t,T)$ is a Gaussian and its mean and variance can be calculated.

$$
\begin{aligned}
\mu_{X_i}(t,T) &\triangleq E_t[X_i(t,T)]\\
&= \left(r-\frac{1}{2}\sigma_{S_i}^2-\hat{\alpha}_i\right)(T-t)+\frac{1}{\kappa_i}(\hat{\alpha}_i-\delta_i(t))(1-e^{-\kappa_i(T-t)}).
\end{aligned}
$$

The covariance $\sigma_{X_1X_2}(t,T)$ is

$$
\begin{aligned}
&\sigma_{X_1X_2}(t,T)\triangleq E_t[(X_1(t,T)-\mu_{X_1}(t,T))(X_2(t,T)-\mu_{X_2}(t,T))]\\
=&\left(\sigma_{S_1S_2}-\frac{\sigma_{S_1\delta_2}}{\kappa_2}-\frac{\sigma_{S_2\delta_1}}{\kappa_1}+\frac{\sigma_{\delta_1\delta_2}}{\kappa_1\kappa_2}\right)(T-t)+\left(\frac{\sigma_{S_1\delta_2}}{\kappa_2^2}-\frac{\sigma_{\delta_1\delta_2}}{\kappa_1\kappa_2^2}\right)(1-e^{-\kappa_2(T-t)})\\
&+\left(\frac{\sigma_{S_2\delta_1}}{\kappa_1^2}-\frac{\sigma_{\delta_1\delta_2}}{\kappa_1^2\kappa_2}\right)(1-e^{-\kappa_1(T-t)})+\frac{\sigma_{\delta_1\delta_2}}{\kappa_1\kappa_2(\kappa_1+\kappa_2)}(1-e^{-(\kappa_1+\kappa_2)(T-t)})
\end{aligned}
$$

and we suppose the correlation $\rho_{X_1X_2}(t,T)$ as

$$
\rho_{X_1X_2}(t,T)\triangleq\frac{\sigma_{X_1X_2}(t,T)}{\sigma_{X_1}(t,T)\sigma_{X_2}(t,T)}
$$

[9] For derivation, see Bjerksund (1991).

Thus $X_i(t,T)$ can be expressed as

$$\boldsymbol{X} \triangleq \begin{pmatrix} X_1(t,T) \\ X_2(t,T) \end{pmatrix} \sim N(\boldsymbol{\mu}_X(t,T), \Sigma_X(t,T))$$

where

$$\begin{aligned} \boldsymbol{\mu}_X(t,T) &\triangleq \begin{pmatrix} \mu_{X_1}(t,T) \\ \mu_{X_2}(t,T) \end{pmatrix} \\ \Sigma_X(t,T) &\triangleq \begin{pmatrix} \sigma^2_{X_1}(t,T) & \sigma_{X_1X_2}(t,T) \\ \sigma_{X_1X_2}(t,T) & \sigma^2_{X_2}(t,T) \end{pmatrix}. \end{aligned}$$

6.2 Proof of Proposition 2.1

We calculate the following equation in this subsection. Let us use the notation $\Phi(\cdot)$ and $\phi(\cdot)$ as the standard normal distribution and density function, respetively, and also $N(\cdot|\mu,\sigma^2)$ and $n(\cdot|\mu,\sigma^2)$ as normal distribution and density function with μ and σ^2 as mean and variance, respectively.

$$S_e(t) = e^{-r(T-t)} E_t[((H_1S_1(T) - H_2S_2(T)) \wedge Z) \vee 0].$$

For notational convenience, we will omit the time parameters such as $\mu_{X_i} = \mu_{X_i}(t,T)$. The expectation is

$$\begin{aligned} & E_t[((H_1S_1(T) - H_2S_2(T)) \wedge Z) \vee 0] \\ = \;& H_1S_1(t) \int_{D_1} \exp\{x_1\} n(\boldsymbol{x}|\boldsymbol{\mu}_X, \Sigma_X) d\boldsymbol{x} - H_2S_2(t) \int_{D_1} \exp\{x_2\} n(\boldsymbol{x}|\boldsymbol{\mu}_X, \Sigma_X) d\boldsymbol{x} \\ & + Z \int_{D_2} n(\boldsymbol{x}|\boldsymbol{\mu}_X, \Sigma_X) d\boldsymbol{x} \end{aligned}$$

where

$$\begin{aligned} d(x_2, Z) &= \ln(H_2S_2(t)\exp\{x_2\} + Z) - \ln(H_1S_1(t)) \\ D_1 &= \{\boldsymbol{x} = [x_1, x_2]^\top | d(x_2, 0) \le x_1 \le d(x_2, Z)\} \\ D_2 &= \{\boldsymbol{x} = [x_1, x_2]^\top | x_1 > d(x_2, Z)\}. \end{aligned}$$

We calculate each integral. Let us use $\boldsymbol{e}_i$ to be the unit vector which i-th element is one. For the integrals of the first and second term, we have

$$\begin{aligned} & \int_{D_1} \exp\{x_i\} n(\boldsymbol{x}|\boldsymbol{\mu}_X, \Sigma_X) d\boldsymbol{x} \\ = \;& \exp\left\{\mu_{X_i} + \frac{1}{2}\sigma^2_{X_i}\right\} \int_{D_1} (2\pi)^{-1} |\Sigma_X|^{-\frac{1}{2}} \\ & \times \exp\left\{ -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_X - \Sigma_X \boldsymbol{e}_i)^\top \Sigma_X^{-1} (\boldsymbol{x} - \boldsymbol{\mu}_X - \Sigma_X \boldsymbol{e}_i) \right\} d\boldsymbol{x} \end{aligned}$$

where we completed the squares. Furthermore, the integral can be expanded by changing the variables.

$$\begin{aligned}
&\int_{D_1} (2\pi)^{-1}|\Sigma_X|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_X-\Sigma_X\boldsymbol{e}_1)^\top\Sigma_X^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_X-\Sigma_X\boldsymbol{e}_1)\right\} d\boldsymbol{x} \\
=& \int_{-\infty}^{\infty}\int_{d_1(x_2,0)}^{d_1(x_2,Z)} (2\pi(1-\rho_{X_1X_2}^2))^{-\frac{1}{2}}\sigma_{X_1}^{-1}\exp\left\{-\frac{y^2}{2}\right\}(1-\rho_{X_1X_2}^2)^{\frac{1}{2}}\sigma_{X_1}dy \\
& \times(2\pi)^{-\frac{1}{2}}\sigma_{X_2}^{-1}\exp\left\{-\frac{1}{2}\left(\frac{x_2-\mu_{X_2}-\sigma_{X_1X_2}}{\sigma_{X_2}}\right)^2\right\}dx_2 \\
=& \int_{-\infty}^{\infty}(\Phi(d_1(x_2,Z))-\Phi(d_1(x_2,0)))n(x_2|\mu_{X_2}+\sigma_{X_1X_2},\sigma_{X_2}^2)dx_2
\end{aligned}$$

where

$$\begin{aligned}
d_1(x,z) &= \frac{\ln(H_2S_2(t)\exp\{x\}+z)-\ln(H_1S_1(t))-\mu_{X_1}-\sigma_{X_1}^2}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} \\
& -\frac{\rho_{X_1X_2}\sigma_{X_1}\frac{x-\mu_{X_2}-\sigma_{X_1X_2}}{\sigma_{X_2}}}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}}
\end{aligned}$$

In addition, we can simplify the second part of the integration. Generally it is known that,

$$\begin{aligned}
\Phi(d_1) &= P(X_1\le d_1)=P(X_1\le d_1,X_2\le\infty) \qquad (7)\\
&= \int_{-\infty}^{\infty}\Phi\left(\frac{d_1-\rho_{12}x_2}{\sqrt{1-\rho_{12}^2}}\right)\phi(x_2)dx_2
\end{aligned}$$

where

$$\begin{gathered}
[X_1,X_2]\sim N(\mathbf{0},\mathbf{\Sigma}) \\
\mathbf{\Sigma}=\begin{bmatrix}1 & \rho_{12}\\ \rho_{12} & 1\end{bmatrix}
\end{gathered}$$

Notice that,

$$\begin{aligned}
d_1(x_2,0) &= \frac{\ln(H_2S_2(t)/H_1S_1(t))-\mu_{X_1}-\sigma_{X_1}^2+\rho_{X_1X_2}\sigma_{X_1}\frac{\mu_{X_2}+\sigma_{X_1X_2}}{\sigma_{X_2}}}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} \\
& -\frac{\left(\frac{\rho_{X_1X_2}\sigma_{X_1}}{\sigma_{X_2}}-1\right)(\sigma_{X_2}\hat{x}_1+\mu_{X_2}+\sigma_{X_1X_2})}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} \\
&= \frac{\hat{\mu}_1-\hat{\rho}\hat{x}_1}{\sqrt{1-\hat{\rho}^2}}
\end{aligned}$$

where we defined $\hat{x}_1 = (x_2 - \mu_{X_2} - \sigma_{X_1X_2})/\sigma_{X_2}$ and used the following facts.

$$
\begin{aligned}
\frac{\hat{\rho}}{\sqrt{1-\hat{\rho}^2}} &\equiv \frac{\left(\frac{\rho_{X_1X_2}\sigma_{X_1}}{\sigma_{X_2}} - 1\right)\sigma_{X_2}}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} = \frac{\rho_{X_1X_2}\sigma_{X_1} - \sigma_{X_2}}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} \\
\Rightarrow \hat{\rho} &= \frac{\rho_{X_1X_2}\sigma_{X_1} - \sigma_{X_2}}{\sqrt{\sigma_{X_1}^2 - 2\rho_{X_1X_2}\sigma_{X_1}\sigma_{X_2} + \sigma_{X_2}^2}} \\
\hat{\mu}_1 &\equiv \sqrt{1-\hat{\rho}^2}\left(\frac{\ln(H_2S_2(t)/H_1S_1(t)) - \mu_{X_1} - \sigma_{X_1}^2 + \rho_{X_1X_2}\sigma_{X_1}\frac{\mu_{X_2}+\sigma_{X_1X_2}}{\sigma_{X_2}}}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}}\right. \\
&\quad \left. - \frac{\left(\frac{\rho_{X_1X_2}\sigma_{X_1}}{\sigma_{X_2}} - 1\right)(\mu_{X_2} + \sigma_{X_1X_2})}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}}\right) \\
&= \frac{\ln(H_2S_2(t)/H_1S_1(t)) - \mu_{X_1} + \mu_{X_2} - \sigma_{X_1}^2 + \sigma_{X_1X_2}}{\sqrt{\sigma_{X_1}^2 - 2\sigma_{X_1X_2} + \sigma_{X_2}^2}}
\end{aligned}
$$

Now, we have

$$
\begin{aligned}
& -\int_{-\infty}^{\infty} \Phi(d_1(x_2,0))n(x_2|\mu_{X_2} + \sigma_{X_1X_2}, \sigma_{X_2}^2)dx_2 \\
= & -\int_{-\infty}^{\infty} \Phi\left(\frac{\hat{\mu}_1 - \hat{\rho}\hat{x}_1}{\sqrt{1-\hat{\rho}^2}}\right)\phi(\hat{x}_1)d\hat{x}_1 = -\Phi(\hat{\mu}_1)
\end{aligned}
$$

where we used $n(x|\mu_{X_2} + \sigma_{X_1X_2}, \sigma_{X_2}^2) = \frac{1}{\sigma_{X_2}}\phi(\frac{x-\mu_{X_2}-\sigma_{X_1X_2}}{\sigma_{X_2}})$, changed the variables and (7).

The other integrals are calculated in similar manner.

$$
\begin{aligned}
& \int_{D_1} (2\pi)^{-1}|\Sigma_X|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu}_X - \Sigma_X\boldsymbol{e}_2)^\top\Sigma_X^{-1}(\boldsymbol{x} - \boldsymbol{\mu}_X - \Sigma_X\boldsymbol{e}_2)\right\}d\boldsymbol{x} \\
= & \int_{-\infty}^{\infty} \Phi(d_2(x_2,Z))n(x_2|\mu_{X_2} + \sigma_{X_2}^2, \sigma_{X_2}^2)dx_2 - \Phi(\hat{\mu}_2)
\end{aligned}
$$

where

$$
\begin{aligned}
d_2(x,z) &= \frac{\ln(H_2S_2(t)\exp\{x\} + z) - \ln(H_1S_1(t)) - \mu_{X_1} - \rho_{X_1X_2}\sigma_{X_1}\frac{x-\mu_{X_2}}{\sigma_{X_2}}}{\sigma_{X_1}\sqrt{1-\rho_{X_1X_2}^2}} \\
\hat{\mu}_2 &= \frac{\ln(H_2S_2(t)/H_1S_1(t)) - \mu_{X_1} + \mu_{X_2} - \sigma_{X_1X_2} + \sigma_{X_2}^2}{\sqrt{\sigma_{X_1}^2 - 2\sigma_{X_1X_2} + \sigma_{X_2}^2}}
\end{aligned}
$$

The integral of the last term is

$$\begin{aligned}
&\int_{D_2} (2\pi)^{-1}|\Sigma_X|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_X)^\top \Sigma_X^{-1}(\boldsymbol{x}-\boldsymbol{\mu}_X)\right\} d\boldsymbol{x} \\
= &\int_{-\infty}^{\infty} \Phi(-d_2(x_2,Z)) n(x_2|\mu_{X_2},\sigma_{X_2}^2) dx_2
\end{aligned}$$

Collecting all terms, we have the valuation formula.

$$S_e(t) = \hat{H}_1(t,T)S_1(t) - \hat{H}_2(t,T)S_2(t) + \hat{H}_3(t,T)Z \tag{8}$$

where

$$\begin{aligned}
\hat{H}_1(t,T) &= H_1 \exp\left\{ -r(T-t) + \mu_{X_1}(t,T) + \frac{1}{2}\sigma_{X_1}^2(t,T)\right\} \\
&\times\left(\int_{-\infty}^{\infty} \Phi(d_1(x_2,Z)) n(x_2|\mu_{X_2}(t,T)+\sigma_{X_1X_2}(t,T), \sigma_{X_2}^2(t,T)) dx_2 \right. \\
&\left. -\Phi(\hat{\mu}_1(t,T)) \right)
\end{aligned}$$

$$\begin{aligned}
\hat{H}_2(t,T) &= H_2 \exp\left\{ -r(T-t) + \mu_{X_2}(t,T) + \frac{1}{2}\sigma_{X_2}^2(t,T)\right\} \\
&\times\left(\int_{-\infty}^{\infty} \Phi(d_2(x_2,Z)) n(x_2|\mu_{X_2}(t,T)+\sigma_{X_2}^2(t,T), \sigma_{X_2}^2(t,T)) dx_2 \right. \\
&\left. -\Phi(\hat{\mu}_2(t,T)) \right)
\end{aligned}$$

$$\begin{aligned}
\hat{H}_3(t,T) &= \exp(-r(T-t)) \int_{-\infty}^{\infty} \Phi(-d_2(x_2,Z)) n(x_2|\mu_{X_2}(t,T),\sigma_{X_2}^2(t,T)) dx_2 \\
d_1(x,z) &= d_2(x,z) - \sigma_{X_1}(t,T)\sqrt{1-\rho_{X_1X_2}^2(t,T)} \\
d_2(x,z) &= \frac{\ln(H_2S_2(t)\exp\{x\}+z) - \ln(H_1S_1(t)) - \mu_{X_1}(t,T)}{\sigma_{X_1}(t,T)\sqrt{1-\rho_{X_1X_2}^2(t,T)}} \\
&\quad -\frac{\rho_{X_1X_2}(t,T)\sigma_{X_1}(t,T)\frac{x-\mu_{X_2}(t,T)}{\sigma_{X_2}(t,T)}}{\sigma_{X_1}(t,T)\sqrt{1-\rho_{X_1X_2}^2(t,T)}}
\end{aligned}$$

$$\hat{\mu}_1(t,T) = \frac{\ln(H_2S_2(t)/H_1S_1(t)) - \mu_{X_1}(t,T) + \mu_{X_2}(t,T) - \sigma^2_{X_1}(t,T) + \sigma_{X_1X_2}(t,T)}{\sqrt{\sigma^2_{X_1}(t,T) - 2\sigma_{X_1X_2}(t,T) + \sigma^2_{X_2}(t,T)}}$$

$$\hat{\mu}_2(t,T) = \hat{\mu}_1(t,T) + \sqrt{\sigma^2_{X_1}(t,T) - 2\sigma_{X_1X_2}(t,T) + \sigma^2_{X_2}(t,T)}$$

6.3 Proof of Proposition 3.1

In this subsection, we derive the hedging strategy for emission allowance using futures commodities. First, we use the future commodity prices equation written in terms of spot commodity prices and derive the future price process using Ito's lemma. This price process can be explicitly written in terms of futures price levels. Then, we calculate the expectation and covariance of stochastic terms of futures price using properties of stochastic calculus.

$$G_i(t,T) = S_i(t)e^{\mu_{\hat{X}_i}(t,T) + \frac{\sigma^2_{\hat{X}_i}(t,T)}{2}}$$

where

$$\begin{aligned}
\mu_{\hat{X}_i}(t,T) &= E_t[\hat{X}_i(t,T)] \\
&= \left(r - \frac{\sigma^2_{S_i}}{2} - \hat{\alpha}_i\right)(T-t) + \frac{(\hat{\alpha}_i - \delta_i(t))}{\kappa_i}(1 - e^{-\kappa_i(T-t)})
\end{aligned}$$

and

$$\begin{aligned}
\sigma^2_{\hat{X}_i}(t,T) &= E_t[(\hat{X}_i(t,T) - \mu_{\hat{X}_i}(t,T))^2] \\
&= \left(\sigma^2_{S_i} + \frac{\sigma^2_{\delta_i}}{\kappa_i^2} - \frac{2\sigma_{S_i\delta_i}}{\kappa_i}\right)(T-t) + \frac{\sigma^2_{\delta_i}}{2\kappa_i^3}(1 - e^{-2\kappa_i(T-t)}) \\
&\quad +2\left(-\frac{\sigma^2_{\delta_i}}{\kappa_i^3} + \frac{\sigma_{S_i\delta_i}}{\kappa_i^2}\right)(1 - e^{-\kappa_i(T-t)})
\end{aligned}$$

We need the following partial derivatives.

$$\begin{aligned}
\frac{\partial G_i(t,T)}{\partial S_i(t)} &= \frac{G_i(t,T)}{S_i(t)}, \\
\frac{\partial G_i(t,T)}{\partial \delta_i(t)} &= -\frac{1 - e^{-\kappa_i(T-t)}}{\kappa_i} G_i(t,T).
\end{aligned}$$

Since the futures price $G_i(t,T)$ is a function of $S_i(t), \delta_i(t)$ and twice differentiable, we can use the Ito's lemma and the dynamics of future price is

$$\begin{aligned} dG_i(t,T) &= \sigma_{S_i} S_i(t) \frac{\partial G_i}{\partial S_i} dW_{S_i}(t) + \sigma_{\delta_i} \frac{\partial G_i}{\partial \delta_i} dW_{\delta_i}(t) \\ &= \left(\sigma_{S_i} dW_{S_i}(t) - \sigma_{\delta_i} \frac{1 - e^{-\kappa_i (T-t)}}{\kappa_i} dW_{\delta_i}(t) \right) G_i(t,T). \end{aligned}$$

where the drift term is 0 since $G_i(t,T)$ is martingale under the risk-neutral probability.

Again, using Ito's lemma we have,

$$\begin{aligned} & d \log G_i(t,T) \\ = & -\frac{1}{2} \left[\sigma_{S_i}^2 - 2\sigma_{S_i \delta_i} \left(\frac{1 - e^{-\kappa_i (T-t)}}{\kappa_i} \right) + \sigma_{\delta_i}^2 \left(\frac{1 - 2e^{-\kappa_i (T-t)} + e^{-2\kappa_i (T-t)}}{\kappa_i^2} \right) \right] dt \\ & + \sigma_{S_i} dW_{S_i}(t) - \sigma_{\delta_i} \frac{1 - e^{-\kappa_i (T-t)}}{\kappa_i} dW_{\delta_i}(t) \end{aligned}$$

The futures price can be expressed as follows.

$$G_i(T_0, T_i) = G_i(t, T_i) e^{\hat{X}_{G_i}(t, T_0, T_i)}, \quad t \le T_0 \le T_i$$

where

$$\begin{aligned} \hat{X}_{G_i}(t, T_0, T_i) &\equiv \mu_{\hat{X}_{G_i}}(t, T_0, T_i) \\ & \quad + \int_t^{T_0} \sigma_{S_i} dW_{S_i}(u) - \int_t^{T_0} \sigma_{\delta_i} \frac{1 - e^{-\kappa_i (T_i - u)}}{\kappa_i} dW_{\delta_i}(u) \end{aligned}$$

The expectation value $\mu_{\hat{X}_{G_i}}(t, T_0, T_i)$ is

$$\begin{aligned} & \mu_{\hat{X}_{G_i}}(t, T_0, T_i) \equiv E_t[\hat{X}_{G_i}(t, T_0, T_i)] \\ = & -\frac{1}{2} \left[\sigma_{S_i}^2 (T_0 - t) - 2\sigma_{S_i \delta_i} \frac{1}{\kappa_i} \left\{ (T_0 - t) - \frac{1}{\kappa_i} (e^{-\kappa_i (T_i - T_0)} - e^{-\kappa_i (T_i - t)}) \right\} \right. \\ & \left. + \sigma_{\delta_i}^2 \left\{ \frac{T_0 - t}{\kappa_i^2} - \frac{2}{\kappa_i^3} (e^{-\kappa_i (T_i - T_0)} - e^{-\kappa_i (T_i - t)}) + \frac{1}{2\kappa_i^3} (e^{-2\kappa_i (T_i - T_0)} - e^{-2\kappa_i (T_i - t)}) \right\} \right] \end{aligned}$$

The covariance of $\hat{X}_{G_i}(t,T_0,T_i)$ and $\hat{X}_{G_j}(t,T_0,T_j)$ is calculated as follows.

$$
\begin{aligned}
&\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}}(t,T_0,T_i,T_j) \equiv \mathrm{cov}_t[\hat{X}_{G_i}(t,T_0,T_i),\hat{X}_{G_j}(t,T_0,T_j)] \\
= \;& \sigma_{S_iS_j}(T_0-t) - \sigma_{S_i\delta_j}\frac{1}{\kappa_j}\left\{(T_0-t) - \frac{1}{\kappa_j}(e^{-\kappa_j(T_j-T_0)} - e^{-\kappa_j(T_j-t)})\right\} \\
& -\sigma_{\delta_iS_j}\frac{1}{\kappa_i}\left\{(T_0-t) - \frac{1}{\kappa_i}(e^{-\kappa_i(T_i-T_0)} - e^{-\kappa_i(T_i-t)})\right\} \\
& +\sigma_{\delta_i\delta_j}\left[\frac{1}{\kappa_i\kappa_j}\left\{(T_0-t) - \frac{1}{\kappa_i}(e^{-\kappa_i(T_i-T_0)} - e^{-\kappa_i(T_i-t)}) - \frac{1}{\kappa_j}(e^{-\kappa_j(T_j-T_0)} - e^{-\kappa_j(T_j-t)})\right.\right. \\
& \left.\left. +\frac{1}{\kappa_i+\kappa_j}(e^{-\kappa_i(T_i-T_0)-\kappa_j(T_j-T_0)} - e^{-\kappa_i(T_i-t)-\kappa_j(T_j-t)})\right\}\right]
\end{aligned}
$$

Now we derive the emission allowance futures price using commodity future prices.

$$
\begin{aligned}
G_e(t,T) &= E_t[S_e(T)] = E_t[((H_1S_1(T) - H_2S_2(T)) \wedge Z) \vee 0] \\
&= E_t[((H_1G_1(t,T)e^{\hat{X}_{G_1}(t,T,T)} - H_2G_2(t,T)e^{\hat{X}_{G_2}(t,T,T)}) \wedge Z) \vee 0]
\end{aligned}
$$

With the same argument as in the proof of Proposition 2.1, we have

$$
G_e(t,T) = \hat{\hat{H}}_1(t,T)G_1(t,T) - \hat{\hat{H}}_2(t,T)G_2(t,T) + \hat{\hat{H}}_3(t,T)Z
$$

where

$$
\begin{aligned}
\hat{\hat{H}}_1(t,T) = \;& H_1\exp\left\{\mu_{\hat{X}_{G_1}}(t,T,T) + \frac{1}{2}\sigma^2_{\hat{X}_{G_1}}(t,T,T,T)\right\} \\
& \times\left\{\int_{-\infty}^{\infty}\Phi(d_{G_1}(x_2,Z))\right. \\
& \times n(x_2|\mu_{\hat{X}_{G_2}}(t,T,T) + \sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T), \sigma^2_{\hat{X}_{G_2}}(t,T,T,T))dx_2 \\
& \left. -\Phi(\hat{\mu}_{G_1}(t,T))\right\}
\end{aligned}
$$

$$\begin{aligned}
\hat{\hat{H}}_2(t,T) &= H_2 \exp\left\{\mu_{\hat{X}_{G_2}}(t,T,T) + \frac{1}{2}\sigma^2_{\hat{X}_{G_2}}(t,T,T,T)\right\} \\
&\quad \times\left\{ \int_{-\infty}^{\infty} \Phi(d_{G_2}(x_2,Z)) \right. \\
&\quad \times n(x_2|\mu_{\hat{X}_{G_2}}(t,T,T) + \sigma^2_{\hat{X}_{G_2}}(t,T,T,T), \sigma^2_{\hat{X}_{G_2}}(t,T,T,T))dx_2 \\
&\quad \left. -\Phi(\hat{\mu}_{G_2}(t,T)) \right\} \\
\hat{\hat{H}}_3(t,T) &= \int_{-\infty}^{\infty} \Phi(-d_{G_2}(x_2,Z))n(x_2|\mu_{\hat{X}_{G_2}}(t,T,T), \sigma^2_{\hat{X}_{G_2}}(t,T,T,T))dx_2 \\
d_{G_1}(x,Z) &= d_{G_2}(x,Z) - \sigma_{\hat{X}_{G_1}}(t,T,T,T)\sqrt{1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T)} \\
d_{G_2}(x,Z) &= \frac{\ln(H_2G_2(t,T)\exp(x) + Z) - \ln(H_1G_1(t,T)) - \mu_{\hat{X}_{G_1}}(t,T,T)}{\sigma_{\hat{X}_{G_1}}(t,T,T,T)\sqrt{1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T)}} \\
&\quad - \frac{\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T)\sigma_{\hat{X}_{G_1}}(t,T,T,T)\frac{x-\mu_{\hat{X}_{G_2}}(t,T,T)}{\sigma_{\hat{X}_{G_2}}(t,T,T,T)}}{\sigma_{\hat{X}_{G_1}}(t,T,T,T)\sqrt{1-\rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T)}} \\
\hat{\mu}_{G_1}(t,T) &= \frac{\ln(H_2G_2(t,T)/H_1G_1(t,T)) - \mu_{\hat{X}_{G_1}}(t,T,T) + \mu_{\hat{X}_{G_2}}(t,T,T)}{\sqrt{\sigma^2_{\hat{X}_{G_1}}(t,T,T,T) - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T) + \sigma^2_{\hat{X}_{G_2}}(t,T,T,T)}} \\
&\quad + \frac{-\sigma^2_{\hat{X}_{G_1}}(t,T,T,T) + \sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T)}{\sqrt{\sigma^2_{\hat{X}_{G_1}}(t,T,T,T) - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T) + \sigma^2_{\hat{X}_{G_2}}(t,T,T,T)}} \\
\hat{\mu}_{G_2}(t,T) &= \hat{\mu}_{G_1}(t,T) + \sqrt{\sigma^2_{\hat{X}_{G_1}}(t,T,T,T) - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}(t,T,T,T) + \sigma^2_{\hat{X}_{G_2}}(t,T,T,T)}
\end{aligned}$$

We now derive the hedging equation for emission allowance futures price using commodity futures prices. Using Ito's lemma, the dynamics of emission allowance futures price $dG_e(t,T)$ is

$$\begin{aligned}
dG_e(t,T) &= G_1(t,T)d\hat{\hat{H}}_1(t,T) + \hat{\hat{H}}_1(t,T)dG_1(t,T) + d\hat{\hat{H}}_1(t,T)dG_1(t,T) \\
&\quad -G_2(t,T)d\hat{\hat{H}}_2(t,T) - \hat{\hat{H}}_2(t,T)dG_2(t,T) - d\hat{\hat{H}}_2(t,T)dG_2(t,T) \\
&\quad +Zd\hat{\hat{H}}_3(t,T)
\end{aligned}$$

and $d\hat{\hat{H}}_i(t,T)$ is

$$\begin{aligned} d\hat{\hat{H}}_i(t,T) &= \frac{\partial \hat{\hat{H}}_i(t,T)}{\partial t}dt + \sum_{j=1}^{2} \frac{\partial \hat{\hat{H}}_i(t,T)}{\partial G_j(t,T)} dG_j(t,T) \\ &+ \frac{1}{2}\sum_{j,k=1}^{2} \frac{\partial^2 \hat{\hat{H}}_i(t,T)}{\partial G_j(t,T)\partial G_k(t,T)} \Bigg(\sigma_{S_j}\sigma_{S_k} G_j(t,T)G_k(t,T) \\ &- \sigma_{S_j\delta_k} \frac{1-e^{-\kappa_k(T-t)}}{\kappa_k} G_j(t,T)G_k(t,T) \\ &- \sigma_{S_k\delta_j} \frac{1-e^{-\kappa_j(T-t)}}{\kappa_j} G_j(t,T)G_k(t,T) \\ &+ \sigma_{\delta_j\delta_k} \frac{(1-e^{-\kappa_j(T-t)})(1-e^{-\kappa_k(T-t)})}{\kappa_j\kappa_k} G_j(t,T)G_k(t,T) \Bigg) dt \end{aligned}$$

Substituting $d\hat{\hat{H}}_i(t,T)$ to $dG_e(t,T)$, we have

$$\begin{aligned} & dG_e(t,T) \\ = & \Bigg\{ \frac{\partial \hat{\hat{H}}_1(t,T)}{\partial t} G_1(t,T) - \frac{\partial \hat{\hat{H}}_2(t,T)}{\partial t} G_2(t,T) + Z \frac{\partial \hat{\hat{H}}_3(t,T)}{\partial t} \\ & + \sum_{j,k=1}^{2} \Bigg(\frac{G_1(t,T)}{2} \frac{\partial^2 \hat{\hat{H}}_1(t,T)}{\partial G_j(t,T)\partial G_k(t,T)} - \frac{G_2(t,T)}{2} \frac{\partial^2 \hat{\hat{H}}_2(t,T)}{\partial G_j(t,T)\partial G_k(t,T)} \\ & + \frac{Z}{2} \frac{\partial^2 \hat{\hat{H}}_3(t,T)}{\partial G_j(t,T)\partial G_k(t,T)} \Bigg) \Bigg(\sigma_{S_jS_k} G_j(t,T)G_k(t,T) \\ & - \sigma_{S_j\delta_k} \frac{1-e^{-\kappa_k(T-t)}}{\kappa_k} G_j(t,T)G_k(t,T) - \sigma_{S_k\delta_j} \frac{1-e^{-\kappa_j(T-t)}}{\kappa_j} G_j(t,T)G_k(t,T) \\ & + \sigma_{\delta_j\delta_k} \frac{(1-e^{-\kappa_j(T-t)})(1-e^{-\kappa_k(T-t)})}{\kappa_j\kappa_k} G_j(t,T)G_k(t,T) \Bigg) \\ & + \frac{\partial \hat{\hat{H}}_1(t,T)}{\partial G_1(t,T)} \Bigg(\sigma_{S_1}^2 G_1^2(t,T) - 2\sigma_{S_1\delta_1} \frac{1-e^{-\kappa_1(T-t)}}{\kappa_1} G_1^2(t,T) \\ & + \sigma_{\delta_1}^2 \frac{(1-e^{-\kappa_1(T-t)})^2}{\kappa_1^2} G_1^2(t,T) \Bigg) \end{aligned}$$

$$+\frac{\partial\hat{\hat{H}}_1(t,T)}{\partial G_2(t,T)}\Bigg(\sigma_{S_1S_2}G_1(t,T)G_2(t,T)-\sigma_{S_1\delta_2}\frac{1-e^{-\kappa_2(T-t)}}{\kappa_2}G_1(t,T)G_2(t,T)$$
$$-\sigma_{S_2\delta_1}\frac{1-e^{-\kappa_1(T-t)}}{\kappa_1}G_1(t,T)G_2(t,T)$$
$$+\sigma_{\delta_1\delta_2}\frac{(1-e^{-\kappa_1(T-t)})(1-e^{-\kappa_2(T-t)})}{\kappa_1\kappa_2}G_1(t,T)G_2(t,T)\Bigg)$$
$$-\frac{\partial\hat{\hat{H}}_2(t,T)}{\partial G_2(t,T)}\Bigg(\sigma_{S_2}^2G_2^2(t,T)-2\sigma_{S_2\delta_2}\frac{1-e^{-\kappa_2(T-t)}}{\kappa_2}G_2^2(t,T)$$
$$+\sigma_{\delta_2}^2\frac{(1-e^{-\kappa_2(T-t)})^2}{\kappa_2^2}G_2^2(t,T)\Bigg)$$
$$-\frac{\partial\hat{\hat{H}}_2(t,T)}{\partial G_1(t,T)}\Bigg(\sigma_{S_1S_2}G_1(t,T)G_2(t,T)-\sigma_{S_1\delta_2}\frac{1-e^{-\kappa_2(T-t)}}{\kappa_2}G_1(t,T)G_2(t,T)$$
$$-\sigma_{S_2\delta_1}\frac{1-e^{-\kappa_1(T-t)}}{\kappa_1}G_1(t,T)G_2(t,T)$$
$$+\sigma_{\delta_1\delta_2}\frac{(1-e^{-\kappa_1(T-t)})(1-e^{-\kappa_2(T-t)})}{\kappa_1\kappa_2}G_1(t,T)G_2(t,T)\Bigg)\Bigg\}dt$$
$$+\sum_{j=1}^{2}\frac{\partial\hat{\hat{H}}_1(t,T)}{\partial G_j(t,T)}G_1(t,T)dG_j(t,T)+\hat{\hat{H}}_1(t,T)dG_1(t,T)$$
$$-\sum_{j=1}^{2}\frac{\partial\hat{\hat{H}}_2(t,T)}{\partial G_j(t,T)}G_2(t,T)dG_j(t,T)-\hat{\hat{H}}_2(t,T)dG_2(t,T)$$
$$+\sum_{j=1}^{2}\frac{\partial\hat{\hat{H}}_3(t,T)}{\partial G_j(t,T)}ZdG_j(t,T)$$

Here after, we omit the time parameters. The partial derivatives are calculated as follows.

$$\begin{aligned}\frac{\partial\hat{\hat{H}}_i}{\partial G_j} &= H_i\exp\left\{\mu_{\hat{X}_{G_i}}+\frac{1}{2}\sigma^2_{\hat{X}_{G_i}}\right\}\\ &\quad\times\Bigg\{\int_{-\infty}^{\infty}\phi(d_{G_i}(x_2,Z))\frac{\partial d_{G_i}(x_2,Z)}{\partial G_j}n(x_2|\mu_{\hat{X}_{G_2}}+\sigma_{\hat{X}_{G_2}\hat{X}_{G_i}},\sigma^2_{\hat{X}_{G_2}})dx_2\\ &\quad-\phi(\hat{\mu}_{G_i})\frac{\partial\hat{\mu}_{G_i}}{\partial G_j}\Bigg\},i,j=1,2\end{aligned}$$

$$
\begin{aligned}
\frac{\partial^2 \hat{\hat{H}}_i}{\partial G_j \partial G_k} &= H_i \exp\left\{ \mu_{\hat{X}_{G_i}} + \frac{1}{2}\sigma^2_{\hat{X}_{G_i}} \right\} \\
&\times \Bigg\{ \int_{-\infty}^{\infty} \phi'(d_{G_i}(x_2, Z)) \frac{\partial d_{G_i}(x_2, Z)}{\partial G_k} \frac{\partial d_{G_i}(x_2, Z)}{\partial G_j} \\
&\times n(x_2 | \mu_{\hat{X}_{G_2}} + \sigma_{\hat{X}_{G_2}\hat{X}_{G_i}}, \sigma^2_{\hat{X}_{G_2}}) dx_2 \\
&+ \int_{-\infty}^{\infty} \phi(d_{G_i}(x_2, Z)) \frac{\partial^2 d_{G_i}(x_2, Z)}{\partial G_j \partial G_k} n(x_2 | \mu_{\hat{X}_{G_2}} + \sigma_{\hat{X}_{G_2}\hat{X}_{G_i}}, \sigma^2_{\hat{X}_{G_2}}) dx_2 \\
&- \phi'(\hat{\mu}_{G_i}) \frac{\partial \hat{\mu}_{G_i}}{\partial G_k} \frac{\partial \hat{\mu}_{G_i}}{\partial G_j} - \phi(\hat{\mu}_{G_i}) \frac{\partial^2 \hat{\mu}_{G_i}}{\partial G_j \partial G_k} \Bigg\} \\
\frac{\partial \hat{\hat{H}}_i}{\partial t} &= H_i \left(\frac{\partial \mu_{\hat{X}_{G_i}}}{\partial t} + \frac{1}{2} \frac{\partial \sigma^2_{\hat{X}_{G_i}}}{\partial t} \right) \exp\left\{ \mu_{\hat{X}_{G_i}} + \frac{1}{2}\sigma^2_{\hat{X}_{G_i}} \right\} \\
&\times \Bigg\{ \int_{-\infty}^{\infty} \Phi(d_{G_i}(x_2, Z)) n(x_2 | \mu_{\hat{X}_{G_2}} + \sigma_{\hat{X}_{G_2}\hat{X}_{G_i}}, \sigma^2_{\hat{X}_{G_2}}) dx_2 - \Phi(\hat{\mu}_{G_i}) \Bigg\} \\
&+ H_i \exp\left\{ \mu_{\hat{X}_{G_i}} + \frac{1}{2}\sigma^2_{\hat{X}_{G_i}} \right\} \\
&\times \Bigg\{ \int_{-\infty}^{\infty} \phi(d_{G_i}(x_2, Z)) \frac{\partial d_{G_i}(x_2, Z)}{\partial t} n(x_2 | \mu_{\hat{X}_{G_2}} + \sigma_{\hat{X}_{G_2}\hat{X}_{G_i}}, \sigma^2_{\hat{X}_{G_2}}) dx_2 \\
&+ \int_{-\infty}^{\infty} \Phi(d_{G_i}(x_2, Z)) \frac{\partial n(x_2 | \mu_{\hat{X}_{G_2}} + \sigma_{\hat{X}_{G_2}\hat{X}_{G_i}}, \sigma^2_{\hat{X}_{G_2}})}{\partial t} dx_2 \\
&- \phi(\hat{\mu}_{G_i}) \frac{\partial \hat{\mu}_{G_i}}{\partial t} \Bigg\}, i = 1, 2
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial \hat{\hat{H}}_3}{\partial t} &= -\int_{-\infty}^{\infty} \phi(-d_{G_2}(x_2, Z)) \frac{\partial d_{G_2}(x_2, Z)}{\partial t} n(x_2 | \mu_{\hat{X}_{G_2}}, \sigma^2_{\hat{X}_{G_2}}) dx_2 \\
&+ \int_{-\infty}^{\infty} \Phi(-d_{G_2}(x_2, Z)) \frac{\partial n(x_2 | \mu_{\hat{X}_{G_2}}, \sigma^2_{\hat{X}_{G_2}})}{\partial t} dx_2 \\
\frac{\partial \hat{\hat{H}}_3}{\partial G_j} &= -\int_{-\infty}^{\infty} \phi(-d_{G_2}(x_2, Z)) \frac{\partial d_{G_2}(x_2, Z)}{\partial G_j} n(x_2 | \mu_{\hat{X}_{G_2}}, \sigma^2_{\hat{X}_{G_2}}) dx_2 \\
\frac{\partial \hat{\hat{H}}_3}{\partial G_j \partial G_k} &= \int_{-\infty}^{\infty} \phi'(-d_{G_2}(x_2, Z)) \frac{\partial d_{G_2}(x_2, Z)}{\partial G_k} \frac{\partial d_{G_2}(x_2, Z)}{\partial G_j} n(x_2 | \mu_{\hat{X}_{G_2}}, \sigma^2_{\hat{X}_{G_2}}) dx_2 \\
&- \int_{-\infty}^{\infty} \phi(-d_{G_2}(x_2, Z)) \frac{\partial^2 d_{G_2}(x_2, Z)}{\partial G_j \partial G_k} n(x_2 | \mu_{\hat{X}_{G_2}}, \sigma^2_{\hat{X}_{G_2}}) dx_2
\end{aligned}
$$

where

$$\begin{aligned}
\frac{\partial \mu_{\hat{X}_{G_i}}}{\partial t} &= -\frac{1}{2}\Bigg\{ -\sigma_{S_i}^2 - 2\sigma_{S_i\delta_i}\frac{-1+e^{-\kappa_i(T-t)}}{\kappa_i} \\
&\quad +\sigma_{\delta_i}^2\Bigg(-\frac{1}{\kappa_i^2} + \frac{2}{\kappa_i^2}e^{-\kappa_i(T-t)} - \frac{1}{\kappa_i^2}e^{-2\kappa_i(T-t)}\Bigg)\Bigg\}, i=1,2 \\
\frac{\partial \sigma_{\hat{X}_{G_i}\hat{X}_{G_j}}}{\partial t} &= -\sigma_{S_iS_j} - \frac{\sigma_{S_i\delta_j}}{\kappa_j}(-1+e^{-\kappa_j(T-t)}) - \frac{\sigma_{\delta_iS_j}}{\kappa_i}(-1+e^{-\kappa_i(T-t)}) \\
&\quad +\sigma_{\delta_i\delta_j}\Bigg\{\frac{1}{\kappa_i\kappa_j}(-1+e^{-\kappa_i(T-t)}+e^{-\kappa_j(T-t)}-e^{-(\kappa_i+\kappa_j)(T-t)})\Bigg\}, i,j=1,2 \\
\frac{\partial \rho_{\hat{X}_{G_i}\hat{X}_{G_j}}}{\partial t} &= \frac{\frac{\partial \sigma_{\hat{X}_{G_i}\hat{X}_{G_j}}}{\partial t} - \rho_{\hat{X}_{G_i}\hat{X}_{G_j}}\Bigg(\frac{\partial \sigma_{\hat{X}_{G_i}}}{\partial t}\sigma_{\hat{X}_{G_j}} + \sigma_{\hat{X}_{G_i}}\frac{\partial \sigma_{\hat{X}_{G_j}}}{\partial t}\Bigg)}{\sigma_{\hat{X}_{G_i}}\sigma_{\hat{X}_{G_j}}}, i,j=1,2 \\
\frac{\partial \sigma_{\hat{X}_{G_i}}}{\partial t} &= \frac{1}{2}\sigma_{\hat{X}_{G_i}}^{-1}\frac{\partial \sigma_{\hat{X}_{G_i}}^2}{\partial t}, i=1,2 \\
\frac{\partial d_{G_2}(x_2,Z)}{\partial t} &= (\sigma_{\hat{X}_{G_1}}^2(1-\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}^2))^{-1} \\
&\quad \times\Bigg\{\Bigg(-\frac{\partial \mu_{\hat{X}_{G_1}}}{\partial t} + \frac{-\frac{\partial \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}}{\partial t}\sigma_{\hat{X}_{G_1}}\sigma_{\hat{X}_{G_2}}(x_2-\mu_{\hat{X}_{G_2}})}{\sigma_{\hat{X}_{G_2}}^2} \\
&\quad -\frac{\frac{\partial \sigma_{\hat{X}_{G_1}}}{\partial t}\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}\sigma_{\hat{X}_{G_2}}(x_2-\mu_{\hat{X}_{G_2}})}{\sigma_{\hat{X}_{G_2}}^2} \\
&\quad +\frac{\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}\frac{\partial \mu_{\hat{X}_{G_2}}}{\partial t}}{\sigma_{\hat{X}_{G_2}}^2} + \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}\sigma_{\hat{X}_{G_1}}(x_2-\mu_{\hat{X}_{G_2}})\frac{\frac{\partial \sigma_{\hat{X}_{G_2}}}{\partial t}}{\sigma_{\hat{X}_{G_2}}^2}\Bigg) \\
&\quad \times\sqrt{\sigma_{\hat{X}_{G_1}}^2(1-\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}^2)} \\
&\quad -\Bigg(\ln(H_2G_2e^{x_2}+Z) - \ln(H_1G_1) - \mu_{\hat{X}_{G_1}} - \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}\sigma_{\hat{X}_{G_1}}\frac{x_2-\mu_{\hat{X}_{G_2}}}{\sigma_{\hat{X}_{G_2}}}\Bigg) \\
&\quad \times\Bigg(\frac{\partial \sigma_{\hat{X}_{G_1}}}{\partial t}\sqrt{1-\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}^2} - \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}\sigma_{\hat{X}_{G_1}}(1-\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}^2)^{-\frac{1}{2}}\frac{\partial \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}}{\partial t}\Bigg)\Bigg\} \\
\frac{\partial d_{G_1}(x_2,Z)}{\partial t} &= \frac{\partial d_{G_2}(x_2,Z)}{\partial t} - \frac{\partial \sigma_{\hat{X}_{G_1}}}{\partial t}\sqrt{1-\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}^2} \\
&\quad +\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}\sigma_{\hat{X}_{G_1}}(1-\rho_{\hat{X}_{G_1}\hat{X}_{G_2}}^2)^{-\frac{1}{2}}\frac{\partial \rho_{\hat{X}_{G_1}\hat{X}_{G_2}}}{\partial t}
\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \hat{\mu}_{G_1}}{\partial t} &= (\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-1} \\
&\times \Bigg\{ \Bigg(-\frac{\partial \mu_{\hat{X}_{G_1}}}{\partial t} + \frac{\partial \mu_{\hat{X}_{G_2}}}{\partial t} - \frac{\partial \sigma^2_{\hat{X}_{G_1}}}{\partial t} + \frac{\partial \sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}}{\partial t} \Bigg) \\
&\times \sqrt{\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}}} \\
&- (\ln(H_2G_2/H_1G_1) - \mu_{\hat{X}_{G_1}} + \mu_{\hat{X}_{G_2}} - \sigma^2_{\hat{X}_{G_1}} + \sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}) \\
&\times \frac{1}{2}(\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-\frac{1}{2}} \\
&\times \Bigg(\frac{\partial \sigma^2_{\hat{X}_{G_1}}}{\partial t} - 2\frac{\partial \sigma_{\hat{X}_{G_1}\hat{X}_{G_2}}}{\partial t} + \frac{\partial \sigma^2_{\hat{X}_{G_2}}}{\partial t} \Bigg) \Bigg\} \\
\frac{\partial \hat{\mu}_{G_2}}{\partial t} &= (2\pi)^{-\frac{1}{2}} \exp\Bigg\{ -\frac{1}{2}\Bigg(\frac{x_2 - \mu_{\hat{X}_{G_2}} - \sigma_{\hat{X}_{G_2}\hat{X}_{G_i}}}{\sigma_{\hat{X}_{G_2}}} \Bigg)^2 \Bigg\} \sigma^{-1}_{\hat{X}_{G_2}} \\
&\times \Bigg(\frac{x_2 - \mu_{\hat{X}_{G_2}} - \sigma_{\hat{X}_{G_2}\hat{X}_{G_i}}}{\sigma_{\hat{X}_{G_2}}} \Bigg) \\
&\times \frac{\Bigg(\frac{\partial \mu_{\hat{X}_{G_2}}}{\partial t} + \frac{\partial \sigma_{\hat{X}_{G_2}\hat{X}_{G_i}}}{\partial t} \Bigg) \sigma_{\hat{X}_{G_2}} + (x_2 - \mu_{\hat{X}_{G_2}} - \sigma_{\hat{X}_{G_2}\hat{X}_{G_i}}) \frac{\partial \sigma_{\hat{X}_{G_2}}}{\partial t}}{\sigma^2_{\hat{X}_{G_2}}} \\
&- (2\pi)^{-\frac{1}{2}} \exp\Bigg\{ -\frac{1}{2}\Bigg(\frac{x_2 - \mu_{\hat{X}_{G_2}} - \sigma_{\hat{X}_{G_2}\hat{X}_{G_i}}}{\sigma_{\hat{X}_{G_2}}} \Bigg)^2 \Bigg\} \frac{\frac{\partial \sigma_{\hat{X}_{G_2}}}{\partial t}}{\sigma^2_{\hat{X}_{G_2}}}, i = 1, 2 \\
\frac{\partial d_{G_i}(x_2, Z)}{\partial G_1} &= -(\sigma_{\hat{X}_{G_1}} \sqrt{1 - \rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}})^{-1} G_1^{-1}, i = 1, 2 \\
\frac{\partial d_{G_i}(x_2, Z)}{\partial G_2} &= (\sigma_{\hat{X}_{G_1}} \sqrt{1 - \rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}})^{-1} \frac{H_2 e^{x_2}}{H_2 G_2 e^{x_2} + Z}, i = 1, 2 \\
\frac{\partial \hat{\mu}_{G_i}}{\partial G_1} &= -(\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-\frac{1}{2}} G_1^{-1}, i = 1, 2 \\
\frac{\partial \hat{\mu}_{G_i}}{\partial G_2} &= (\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-\frac{1}{2}} G_2^{-1}, i = 1, 2 \\
\phi'(x) &= \frac{-x}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} \\
\frac{\partial^2 d_{G_i}(x_2, Z)}{\partial G_1^2} &= (\sigma_{\hat{X}_{G_1}} \sqrt{1 - \rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}})^{-1} G_1^{-2}, i = 1, 2 \\
\frac{\partial^2 d_{G_i}(x_2, Z)}{\partial G_2^2} &= -(\sigma_{\hat{X}_{G_1}} \sqrt{1 - \rho^2_{\hat{X}_{G_1}\hat{X}_{G_2}}})^{-1} \Bigg(\frac{H_2 e^{x_2}}{H_2 G_2 e^{x_2} + Z} \Bigg)^2, i = 1, 2
\end{aligned}
$$

$$\begin{aligned}
\frac{\partial^2 d_{G_i}(x_2, Z)}{\partial G_1 \partial G_2} &= 0, i = 1, 2 \\
\frac{\partial^2 \hat{\mu}_{G_i}}{\partial G_1^2} &= (\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-\frac{1}{2}} G_1^{-2}, i = 1, 2 \\
\frac{\partial^2 \hat{\mu}_{G_i}}{\partial G_2^2} &= -(\sigma^2_{\hat{X}_{G_1}} - 2\sigma_{\hat{X}_{G_1}\hat{X}_{G_2}} + \sigma^2_{\hat{X}_{G_2}})^{-\frac{1}{2}} G_2^{-2}, i = 1, 2 \\
\frac{\partial^2 \hat{\mu}_{G_i}}{\partial G_1 \partial G_2} &= 0, i = 1, 2 \\
\frac{\partial n(x_2 | \mu_{\hat{X}_{G_2}}, \sigma^2_{\hat{X}_{G_2}})}{\partial t} &= (2\pi)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}\left(\frac{x_2 - \mu_{\hat{X}_{G_2}}}{\sigma_{\hat{X}_{G_2}}}\right)^2 \right\} \sigma^{-1}_{\hat{X}_{G_2}} \\
&\quad \times \left(\frac{x_2 - \mu_{\hat{X}_{G_2}}}{\sigma_{\hat{X}_{G_2}}}\right) \frac{\frac{\partial \mu_{\hat{X}_{G_2}}}{\partial t}\sigma_{\hat{X}_{G_2}} + (x_2 - \mu_{\hat{X}_{G_2}})\frac{\partial \sigma_{\hat{X}_{G_2}}}{\partial t}}{\sigma^2_{\hat{X}_{G_2}}} \\
&\quad -(2\pi)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}\left(\frac{x_2 - \mu_{\hat{X}_{G_2}}}{\sigma_{\hat{X}_{G_2}}}\right)^2 \right\} \frac{\frac{\partial \sigma_{\hat{X}_{G_2}}}{\partial t}}{\sigma^2_{\hat{X}_{G_2}}}
\end{aligned}$$

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