Hitotsubashi ICS-FS Working Paper Series

FS-2013-E-003

Commodity Spread Option with Cointegration

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First version: June 9, 2009

Current version: August 2, 2013

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Abstract

We derive the valuation formula of a European call option and the analytical approximation formula of its American counterpart on the spread of two cointegrated commodity prices, based on the GSC (Gibson-Schwartz with cointegration) model. In the numerical analysis, we compare the spread option values calculated by the GSC model and the GS (Gibson-Schwartz) model that ignores cointegration. Consistent with the intuition that the cointegration prevents the prices from diverging, the GSC model prices the commodity spread option with longer maturity lower than the GS model. Thus, incorporating cointegration is important for valuation and hedging of long-term commodity spread options such as large scale oil refining plant developments.

Keywords: cointegration, commodity prices, convenience yield, energy, spread option.

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1 Introduction

Commodity spread option is an option on a spread of two commodity prices which gives oil energy companies or resource refinery companies a useful tools to hedge their business risk. Examples of commodity spread options are crack spread options on crude oil and petroleum products such as heating oil and spark spread options on electricity and gas. As one may guess, those two commodity prices in the examples may be related to each other. Thus, cointegration is a promising tool, which is yet to be studied in the area of derivative pricing. Indeed, Girma and Paulson (1999) find evidence of cointegration in crude oil, unleaded gasoline, and heating oil. Fezzi and Bunn (2009) examine the relation of electricity, gas, and emission allowances through cointegration. These empirical papers suggest that commodity spread options should be analyzed within the framework of cointegration.

However, few academic literature on commodity spread options have analyzed cointegrated prices. Wilcox (1990), Shimko (1994), Pearson (1995), and Nakajima and Maeda (2007) all have their distinctive characteristics, but cointegration is out of the scope of these papers. Dempster, Medova, and Tang (2008) derived a valuation formula by assuming a spot spread process. But, this is not enough for incorporating cointegration, since their model starts from the spread and does not explicitly consider log commodity prices with unit root process. Thus, a model for commodity spread options is called for that incorporates cointegration, or more generally linear relations, between log commodity prices.

Independently of this paper, Casassus, Liu, and Tang (2013) develop a commodity pricing model with linear relations between commodity prices. To check the validity of their model, they conduct some Monte Carlo simulations to calculate prices of European commodity spread options. However, they do not analytically investigate the valuation of commodity spread option and the effect of cointegration.

In this paper, based on our earlier paper, Nakajima and Ohashi (2012) that incorporate cointegration into a commodity pricing model, we develop a model of commodity spread options with linear relations between commodity prices. We derive a semianalytic formula for European commodity spread options and provide an approximation formula for American commodity spread options. We also investigate properties of spread option prices by conducting sensitivity analyses and show that the commodity spread option price that incorporate cointegration is much lower than the one that ignores cointegration for longer maturity.

More precisely, we use two models to analyze commodity spread options. One is the GS model (1990), which is the benchmark of commodity derivative models. The other is the GSC model (the GS with cointegration model) developed in our preceding paper, Nakajima and Ohashi (2012), which extends the GS model to incorporate linear relations between commodity prices. We derive the valuation formulae for European commodity call spread options from both models. Furthermore, we provide an analytical approximation formula for American call commodity spread options for the GSC model using the framework of Bjerksund and Stensland (1994). Finally, using the parameter values empirically estimated in our previous paper, we conduct a numerical analysis and investigate characteristics of commodity spread options with and without cointegration. The result is simple but critical. If there is cointegration among commodity prices, then the price of commodity spread option with long maturity, such as 3 years or more, using GS model (without cointegration) will be considerably overpriced than that of the GSC model (with cointegration). Since oil related and/or electrical power development projects take a long time until commercial use, the misprice of commodity spread option with long maturity should not be tolerable. Thus, the pricing model of spread options incorporating cointegration such as this one can be a better tool of valuation and risk management for these business than the standard models that ignore cointegration.

This paper is organized as follows: Section 2 formulates the model and derives the valuation formula for European call commodity spread options and the analytical approximation formula for American ones. Section 3 provides the numerical analysis. Section 4 concludes.

2 A Model for Commodity Spread Options

2.1 The Gibson-Schwartz (GS) Model

We first derive the pricing formula of a spread option for the GS model that does not incorporate cointegration. Assume that there are n commodities whose spot prices and convenience yields follow

$$d\ln S_i(t) = \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t)\right)dt + \sigma_{S_i} dW_{S_i}(t), \quad i = 1, 2, \tag{1}$$

$$d\delta_i(t) = \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t), \quad i = 1, 2, \tag{2}$$

under the risk-neutral probability. Here, r is the risk-free interest rate, which is assumed to be constant. Also, σ_{S_i}, κ_i, $\hat{\alpha}_i$, and σ_{δ_i} are constant coefficients. $W(t) = [W_{S_1}(t), ..., W_{S_n}(t), W_{\delta_1}(t), ..., W_{\delta_n}(t)]^\top$ is four-dimensional Brownian motion under the risk-neutral probability with

$$dW_{S_i}(t)dW_{S_j}(t) = \rho_{S_iS_j}dt, dW_{S_i}(t)dW_{\delta_j}(t) = \rho_{S_i\delta_j}dt, dW_{\delta_i}(t)dW_{\delta_j}(t) = \rho_{\delta_i\delta_j}dt, \quad i, j = 1, 2.$$

We show the futures price on commodity i in closed-forms. Note that under the assumptions above, the spot price of commodity i is calculated as[1]

$$\begin{aligned} S_i(T) &= S_i(t)\exp\{\hat{X}_i(t,T)\}, \qquad (3) \\ \hat{X}_i(t,T) &= \left(r - \frac{\sigma_i^2}{2} - \hat{\alpha}_i\right)(T-t) + \frac{(\hat{\alpha}_i - \delta_i(t))}{\kappa_i}(1 - e^{-\kappa_i(T-t)}) \\ &\quad + \sigma_{S_i}(W_{S_i}(T) - W_{S_i}(t)) - \frac{1}{\kappa_i}\sigma_{\delta_i}(W_{\delta_i}(T) - W_{\delta_i}(t)) \\ &\quad + \int_t^T \frac{e^{-\kappa_i(T-s)}}{\kappa_i}\sigma_{\delta_i}dW_{\delta_i}(s). \end{aligned}$$

We denote $E_t[\cdot]$ as expectation under the risk-neutral probability given $\mathcal{F}_t$.[2] Using risk-neutrality and the property of moment generating function, we obtain the futures price of commodity i as follows.

Proposition 2.1. *Assuming (1) and (2), the futures price of commodity i with maturity T at t is given by*

$$\begin{aligned} G_i(t,T) &= E_t[S_i(T)] \\ &= S_i(t)\exp\left\{\mu_{\hat{X}_i}(t,T) + \frac{\sigma^2_{\hat{X}_i}(t,T)}{2}\right\}, \end{aligned}$$

where

$$\begin{aligned} \mu_{\hat{X}_i}(t,T) &= E_t[\hat{X}_i(t,T)] \\ &= \left(r - \frac{\sigma_{S_i}^2}{2} - \hat{\alpha}_i\right)(T-t) + \frac{(\hat{\alpha}_i - \delta_i(t))}{\kappa_i}(1 - e^{-\kappa_i(T-t)}), \end{aligned}$$

[1] See Gibson and Schwartz (1990), Bjerksund (1991), and Schwartz (1997) for derivation.

[2] We assume a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, P)$.

and

$$
\begin{aligned}
\sigma^2_{\hat{X}_i}(t,T) &= E_t[(\hat{X}_i(t,T)-\mu_{\hat{X}_i}(t,T))^2] \\
&= \left(\sigma^2_{S_i}+\frac{\sigma^2_{\delta_i}}{\kappa_i^2}-\frac{2\sigma_{S_i\delta_i}}{\kappa_i}\right)(T-t)+\frac{\sigma^2_{\delta_i}}{2\kappa_i^3}(1-e^{-2\kappa_i(T-t)}) \\
&\quad +2\left(-\frac{\sigma^2_{\delta_i}}{\kappa_i^3}+\frac{\sigma_{S_i\delta_i}}{\kappa_i^2}\right)(1-e^{-\kappa_i(T-t)}).
\end{aligned}
$$

Proof. See Bjerksund (1991). □

A European call commodity spread option between commodities i and j with exercise price K and maturity T_0 is a call option whose payoff at maturity is given by $Max[h_iG_i - h_jG_j - K, 0]$ where h_i and h_j are constants and G_i and G_j are future prices at maturity. The price formula of this spread option is provided in the following proposition.

Proposition 2.2. *Suppose that $S_i(t)$ and $\delta_i(t)$ follow (1) and (2), respectively. The prices of European call commodity spread options at t with exercise price K, where the maturities of the spread option, commodity i, and commodity j futures maturity are T_0, T_i, and T_j, respectively, are given by*

$$
\begin{aligned}
& C^E(K,G_i,G_j,t,T_0,T_i,T_j) \\
= \; & h_iG_i(t,T_i)\exp\{-r(T_0-t)\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_i(x_j,K)) \\
& \times n(x_j|\mu_{\hat{X}_{G_j}GS}(t,T_0,T_j)+\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t,T_0,T_i,T_j),\sigma^2_{\hat{X}_{G_j}GS}(t,T_0,T_j,T_j))dx_j \\
& -h_jG_j(t,T_j)\exp\{-r(T_0-t)\} \\
& \times \int_{-\infty}^{\infty} \Phi(d(x_j,K)) \\
& \times n(x_j|\mu_{\hat{X}_{G_j}GS}(t,T_0,T_j)+\sigma^2_{\hat{X}_{G_j}GS}(t,T_0,T_j,T_j),\sigma^2_{\hat{X}_{G_j}GS}(t,T_0,T_j,T_j))dx_j \\
& -Ke^{-r(T_0-t)}\int_{-\infty}^{\infty}\Phi(d(x_j,K))n(x_j|\mu_{\hat{X}_{G_j}GS}(t,T_0,T_j),\sigma^2_{\hat{X}_{G_j}GS}(t,T_0,T_j,T_j))dx_j,
\end{aligned}
$$

where

$$d(x_j,K) = -\frac{\ln(h_j G_j(t,T_j)e^{x_j}+K) - \ln(h_i G_i(t,T_i)) - \mu_{\hat{X}_{G_i}GS}(t,t_0,T_i)}{\sigma_{\hat{X}_{G_i}GS}(t,T_0,T_i,T_i)\sqrt{1-\rho^2_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t,T_0,T_i,T_j)}}$$

$$+\frac{\rho_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t,T_0,T_i,T_j)\sigma_{\hat{X}_{G_i}GS}(t,T_0,T_i,T_i)\frac{x_j-\mu_{\hat{X}_{G_j}GS}(t,T_0,T_j)}{\sigma_{\hat{X}_{G_j}GS}(t,T_0,T_i,T_j)}}{\sigma_{\hat{X}_{G_i}GS}(t,T_0,T_i,T_i)\sqrt{1-\rho^2_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t,T_0,T_i,T_j)}},$$

$$d_i(x_j,K) = d(x_j,K) + \frac{\sigma^2_{\hat{X}_{G_i}GS}(t,T_0,T_i,T_i) - \rho^2_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t,T_0,T_i,T_j)\sigma^2_{\hat{X}_{G_i}GS}(t,T_0,T_i,T_i)}{\sigma_{\hat{X}_{G_i}GS}(t,T_0,T_i,T_i)\sqrt{1-\rho^2_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t,T_0,T_i,T_j)}},$$

$$\rho_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t,T_0,T_i,T_j) = \frac{\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t,T_0,T_i,T_j)}{\sigma_{\hat{X}_{G_i}GS}(t,T_0,T_i,T_i)\sigma_{\hat{X}_{G_j}GS}(t,T_0,T_j,T_j)},$$

$$\begin{aligned}
\mu_{\hat{X}_{G_i}GS}(t,T) &= E_t[\hat{X}_{G_i}(t,T)] \\
&= -\frac{1}{2}\Bigg[\sigma^2_{S_i}(T_0-t) \\
&\quad -2\frac{\sigma_{S_i\delta_i}}{\kappa_i}\left\{(T_0-t) - \frac{1}{\kappa_i}(e^{-\kappa_i(T_i-T_0)} - e^{-\kappa_i(T_i-t)})\right\} \\
&\quad +\sigma^2_{\delta_i}\Bigg\{\frac{T_0-t}{\kappa_i^2} - \frac{2}{\kappa_i^3}(e^{-\kappa_i(T_i-T_0)} - e^{-\kappa_i(T_i-t)}) \\
&\quad +\frac{1}{2\kappa_i^3}(e^{-2\kappa_i(T_i-T_0)} - e^{-2\kappa_i(T_i-t)})\Bigg\},
\end{aligned}$$

and

$$\begin{aligned}
\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}GS}(t,T) &= E_t[(\hat{X}_{G_i}(t,T) - \mu_{\hat{X}_{G_i}}(t,T))^2] \\
&= \sigma_{S_iS_j}(T_0 - t) \\
&\quad -\frac{\sigma_{S_i\delta_j}}{\kappa_j}\left\{(T_0 - t) - \frac{1}{\kappa_j}(e^{-\kappa_j(T_j - T_0)} - e^{-\kappa_j(T_j - t)})\right\} \\
&\quad -\frac{\sigma_{\delta_iS_j}}{\kappa_i}\left\{(T_0 - t) - \frac{1}{\kappa_i}(e^{-\kappa_i(T_i - T_0)} - e^{-\kappa_i(T_i - t)})\right\} \\
&\quad +\frac{\sigma_{\delta_i\delta_j}}{\kappa_i\kappa_j}\left\{(T_0 - t) - \frac{1}{\kappa_i}(e^{-\kappa_i(T_i - T_0)} - e^{-\kappa_i(T_i - T_0)})\right. \\
&\quad -\frac{1}{\kappa_j}(e^{-\kappa_j(T_j - T_0)} - e^{-\kappa_j(T_j - t)}) \\
&\quad \left. +\frac{1}{\kappa_i + \kappa_j}(e^{-\kappa_i(T_i - T_0) - \kappa_j(T_j - T_0)} - e^{-\kappa_i(T_i - t) - \kappa_j(T_j - t)})\right\}.
\end{aligned}$$

We abbreviate $\sigma_{\hat{X}_{G_i}\hat{X}_{G_i}GS}(t,T)$ *as* $\sigma^2_{\hat{X}_{G_i}GS}(t,T)$.

Proof. The proof is basically the same as the proof of proposition 2.4 in the Appendix. The only differences in that proof are the drifts $\mu_{\hat{X}_{G_i}}$ and the volatilities $\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}}$ are $\mu_{\hat{X}_{G_i GS}}$ and $\sigma_{\hat{X}_{G_i}\hat{X}_{G_j}GS}$, respectively. □

Note that while Shimko (1994) derives the theoretical price of commodity spread options for futures with the same maturities, we allow them to have different maturities.[3]

2.2 The Gibson-Schwartz with cointegration (GSC) Model

We now introduce the GSC model developed by Nakajima and Ohashi (2012). As mentioned, the GSC model is an extension of the GS model and incorporates linear relations between log commodity prices. More precisely, we

[3]Note also that since futures prices are equal to spot prices at maturity, the prices of European call options on futures spread are equal to those on spot spread when all maturities of the option and the underlying futures, i.e., $T_0 = T_i = T_j$.

assume that there are n commodities whose spot prices and convenience yields follow

$$d\ln S_i(t) = \left(r - \frac{\sigma_{S_i}^2}{2} - \delta_i(t) + b_i z(t)\right) dt + \sigma_{S_i} dW_{S_i}(t), \quad i = 1, \ldots, n, \tag{4}$$

$$d\delta_i(t) = \kappa_i(\hat{\alpha}_i - \delta_i(t))dt + \sigma_{\delta_i} dW_{\delta_i}(t), \quad i = 1, \ldots, n, \tag{5}$$

under the risk-neutral probability.[4] b_i, σ_{S_i}, κ_i, $\hat{\alpha}_i$, and σ_{δ_i} are constant coefficients. $W(t) = [W_{S_1}(t), ..., W_{S_n}(t), W_{\delta_1}(t), ..., W_{\delta_n}(t)]^\top$ is $2n$-dimensional Brownian motion under the risk-neutral probability with

$$dW_{S_i}(t)dW_{S_j}(t) = \rho_{S_iS_j}dt, dW_{S_i}(t)dW_{\delta_j}(t) = \rho_{S_i\delta_j}dt, dW_{\delta_i}(t)dW_{\delta_j}(t) = \rho_{\delta_i\delta_j}dt, \quad i, j = 1, \ldots, n.$$

We assume that the commodity prices are related linearly through

$$z(t) = \mu_z + a_0 t + \sum_{i=1}^{n} a_i \ln S_i(t), \tag{6}$$

where μ_z, a_0, and a_is are constants.[5] Assume that $\ln S_i$ are cointegrated. Then by rearranging the equation as $\ln S_1(t) = (-\mu_z - a_0 t - \sum_{i=2}^{n} a_i \ln S_i(t) + z(t))/a_1$ (if $a_1 \neq 0$), $z(t)$ can be interpreted as an error term, a_i as cointegration vectors, and b_i as the adjustment speed of the error correction term.

We obtain the futures price under the GSC model as follows.

Proposition 2.3. *Assuming (4), (5), (6), the futures price of commodity at T_0, which matures at T_i is*

$$G_i(t, T) = e^{\mu_{X_i}(t,T) + \frac{\sigma_{X_i}^2(t,T)}{2}},$$

[4] Notice that while the GS model only concerns two sets of commodity prices and convenience yields, the GSC model may incorporate n sets of commodity prices and convenience yields linear relations.

[5] Although we treat the case where there is only one linear relation between prices, i.e., the case with one-dimensional $z(t)$, we can extend the model to include several linear relations.

where

$$\begin{aligned}
\beta_{S_i0}(t) &= r - \frac{\sigma_{S_i}^2}{2} + b_i\mu_z + b_i a_0 t, \\
\beta_{S_iS_j} &= b_i a_j, \\
\beta_{S_i\delta_i} &= -1, \\
\beta_{\delta_i 0} &= \kappa_i \hat{\alpha}_i, \\
\beta_{\delta_i\delta_i} &= -\kappa_i, \\
\boldsymbol{\beta}_0(t) &= [\beta_{S_10}(t), \cdots, \beta_{S_n0}(t), \beta_{\delta_10}, \cdots, \beta_{\delta_n0}]^\top, \\
\boldsymbol{\beta} &= \begin{bmatrix} \beta_{S_1S_1} & \cdots & \beta_{S_1S_n} & \beta_{S_1\delta_1} & & 0 \\ \vdots & \ddots & \vdots & & \ddots & \\ \beta_{S_nS_1} & \cdots & \beta_{S_nS_n} & 0 & & \beta_{S_n\delta_n} \\ & & & \beta_{\delta_1\delta_1} & & 0 \\ & \mathbf{0} & & & \ddots & \\ & & & 0 & & \beta_{\delta_n\delta_n} \end{bmatrix},
\end{aligned}$$

$$\begin{aligned}
\mu_{X_i}(t,T) &= E_t[\ln S_i(T)] \\
&= \left[e^{T\boldsymbol{\beta}} \left\{ e^{-t\boldsymbol{\beta}} \boldsymbol{X}(t) + \int_t^T e^{-s\boldsymbol{\beta}} \boldsymbol{\beta}_0(s) ds \right\} \right]_i, \\
\sigma_{X_iX_j}(t,T) &= E_t[(\ln S_i(T) - \mu_{X_i}(t,T))(\ln S_j(T) - \mu_{X_i}(t,T))] \\
&= \left[\int_t^T (e^{(T-s)\boldsymbol{\beta}}) \boldsymbol{\Sigma} (e^{(T-s)\boldsymbol{\beta}})^\top ds \right]_{ij},
\end{aligned}$$

See Nakajima and Ohashi (2012) for the proof. We use the notation for expectation $\mu_{X_{G_i}}(t, T_0, T_i) = E_t[X_{G_i}(t, T_0, T_i)]$ and covariance $\sigma_{X_{G_i}X_{G_j}}(t, T_0, T_i, T_j) = E_t[(X_{G_i}(t, T_0, T_i) - \mu_{X_{G_i}}(t, T_0, T_i))(X_{G_j}(t, T_0, T_j) - \mu_{X_{G_j}}(t, T_0, T_j))]$. These are calculated in the Appendix.

Let us now show the price formula for a European commodity spread option.

Proposition 2.4. *Under the GSC model, the prices of the European call commodity spread option at t with exercise price K, where the maturites of spread option, commodity i, and commodity j futures are T_0, T_i, and T_j,* *respectively, are given by*

$$
\begin{aligned}
& C^E(K, G_i, G_j, t, T_0, T_i, T_j) \\
= \;& h_i G_i(t, T_i) \exp\{-r(T_0 - t)\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_i(x_j, K)) n(x_j | \mu_{X_{G_j}}(t, T_0, T_j) + \sigma_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j), \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j)) dx_j \\
& - h_j G_j(t, T_j) \exp\{-r(T_0 - t)\} \\
& \times \int_{-\infty}^{\infty} \Phi(d(x_j, K)) n(x_j | \mu_{X_{G_j}}(t, T_0, T_j) + \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j), \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j)) dx_j \\
& - K e^{-r(T_0 - t)} \int_{-\infty}^{\infty} \Phi(d(x_j, K)) n(x_j | \mu_{X_{G_j}}(t, T_0, T_j), \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j)) dx_j,
\end{aligned}
$$

where

$$
\begin{aligned}
d(x_j, K) = & - \frac{\ln(h_j e^{x_j} + K) - \ln h_i - \mu_{X_{G_i}}(t, t_0, T_i)}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i) \sqrt{1 - \rho^2_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j)}} \\
& + \frac{\rho_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j) \sigma_{X_{G_i}}(t, T_0, T_i, T_i) \frac{x_j - \mu_{X_{G_j}}(t, T_0, T_j)}{\sigma_{X_{G_j}}(t, T_0, T_i, T_j)}}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i) \sqrt{1 - \rho^2_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j)}}, \\
d_i(x_j, K) = & \; d(x_j, K) + \frac{\sigma^2_{X_{G_i}}(t, T_0, T_i, T_i) - \rho^2_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j) \sigma^2_{X_{G_i}}(t, T_0, T_i, T_i)}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i) \sqrt{1 - \rho^2_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j)}}, \\
\rho_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j) = & \; \frac{\sigma_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j)}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i) \sigma_{X_{G_j}}(t, T_0, T_j, T_j)},
\end{aligned}
$$

and $\mu_{X_{G_i}}(t, T)$ *and* $\sigma_{X_{G_i} X_{G_j}}(t, T)$ *are in the Appendix. We abbreviate* $\sigma_{X_{G_i} X_{G_i}}(t, T)$ *as* $\sigma^2_{X_{G_i}}(t, T)$.

Proof. See the Appendix. □

Since the spread options traded in the actual markets are the American type,[6] we also show an approximation formula for American commodity spread options. The derivation is in the Appendix.

[6]A crack spread option such as heating oil/crude oil and RBOB gasoline/crude oil, which are traded on the NYMEX, are both American types.

Proposition 2.5. *Under the GSC model, the price of the American call commodity spread option at t with exercise price K, where the maturities of spread option, commodity i and commodity j futures, are T_0, T_i, and T_j, respectively, is approximated as follows:*

$$\begin{aligned} & C^A(K, G_i, G_j, t, T_0, T_i, T_j) \\ = \; & C^E(K, G_i, G_j, t, T_0, T_i, T_j) + a^A(K, G_i, G_j, t, T_0, T_i, T_j; B^{c,s}), \end{aligned}$$

where

$$\begin{aligned} & a^A(K, G_i, G_j, t, T_0, T_i, T_j; B^{c,s}) \\ \approx \; & r\Bigg[h_i \int_t^{T_0} \exp\left\{ -r(u-t) + \mu_{X_{G_i}}(t,u,T_i) + \frac{1}{2}\sigma^2_{X_{G_i}}(t,u,T_i,T_i) \right\} \\ & \times \int_{-\infty}^{\infty} \Phi(d_{eep2}(x_j,u,K)) \\ & \times n(x_j | \mu_{X_{G_j}}(t,u,T_j) + \sigma_{X_{G_i}X_{G_j}}(t,u,T_i,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \\ & - h_j \int_t^{T_0} \exp\left\{ -r(u-t) + \mu_{X_{G_j}}(t,u,T_j) + \frac{1}{2}\sigma^2_{X_{G_j}}(t,u,T_j,T_j) \right\} \\ & \times \int_{-\infty}^{\infty} \Phi(d_{eep1}(x_j,u,K)) \\ & \times n(x_j | \mu_{X_{G_j}}(t,u,T_j) + \sigma^2_{X_{G_j}}(t,u,T_j,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \\ & - K \int_t^{T_0} e^{-r(u-t)} \int_{-\infty}^{\infty} \Phi(d_{eep1}(x_j,u,K)) \\ & \times n(x_j | \mu_{X_{G_j}}(t,u,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \Bigg], \end{aligned}$$

and

$$
\begin{aligned}
d_{eep1}(x_j,u,K) &= -\frac{\ln(h_j B_j e^{x_j} + B_K K) - \ln h_i - \mu_{X_{G_i}}(t,u,T_i)}{\sigma_{X_{G_i}}(t,u,T_i)\sqrt{1-\rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)}} \\
&\quad + \frac{\rho_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)\sigma_{X_{G_i}}(t,u,T_i)\frac{x_j-\mu_{X_{G_j}}(t,u,T_j)}{\sigma_{X_{G_j}}(t,u,T_j,T_j)}}{\sigma_{X_{G_i}}(t,u,T_i)\sqrt{1-\rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)}}, \\
d_{eep2}(x_j,u,K) &= d_{eep1}(x_j,u,K) + \frac{(1-\rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j))\sigma^2_{X_{G_i}}(t,u,T_i,T_i)}{\sigma_{X_{G_i}}(t,u,T_i)\sqrt{1-\rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)}}.
\end{aligned}
$$

Proof. See the Appendix. □

The formula is the same as that for the GS model. The only differences are drifts $\mu_{X_{G_i}}$ and volatilities $\sigma_{X_{G_i}X_{G_j}}$ are changed to $\mu_{X_{G_i}GS}$ and $\sigma_{X_{G_i}X_{G_j}GS}$, respectively.

Note that the preceding models such as Shimko (1994) and Nakajima and Maeda (2007) did not consider the linear relations between prices. However, empirical analyses such as those of Malliaris and Urrutia (1996) and Girma and Paulson (1999) show evidence of cointegration. Therefore, the commodity spread option should be priced by incorporating the cointegration or more generally a linear relation between log commodity prices. The valuation formulae for European and American call options that we derive in this subsection incorporate linear relations between commodity prices. These linear relations are interpreted as equilibrium or long-term relationships. Thus, the valuation formulae derived in this subsection reflect the long-term equilibrium in derivative pricing.

Note that a commodity spread option is an option on two commodity prices such that the spread relation is fixed within the contract. However, this fixed spread relation may be different from the linear relation corresponding to, say, cointegration, which we cannot observe and need to estimate. Furthermore, such differences between the spread relation and the linear relation may affect the spread option price. Thus, in order to price a spread option, it is not appropriate to start by directly assuming a stochastic process that a commodity spread must satisfy. It is important to start by formulating the stochastic processes that the commodity prices satisfy with some linear relation, and then to derive the price of the spread option.

3 Numerical Analysis

In this section, we numerically analyze the valuation formula of a European commodity future spread option. We use the following parameter values for crude oil (commodity 1) and heating oil (commodity 2) as the benchmark. Those values are estimated in Nakajima and Ohashi (2012) using the NYMEX data from 1990 to 2010.[7]

$$
\begin{aligned}
&G_1(t,T_1) = 35, G_2(t,T_2) = 100,\\
&h_1 = 1, h_2 = 0.42,\\
&\sigma_{S_1} = 0.381896, \sigma_{S_2} = 0.406307, \sigma_{\delta_1} = 0.287109, \sigma_{\delta_2} = 0.699693,\\
&\rho_{S_1S_2} = 0.748660, \rho_{S_1\delta_1} = 0.767305, \rho_{S_1\delta_2} = 0.000072,\\
&\rho_{S_2\delta_1} = 0.628424, \rho_{S_2\delta_2} = 0.620154,\\
&\rho_{\delta_1\delta_2} = 0.165843,\\
&a_1 = -1.187431, a_2 = 1.000000, b_1 = -0.052615, b_2 = -0.356252,\\
&\kappa_1 = 1.140883, \kappa_2 = 1.085038,\\
&T_0 = 1250/250, T_1 = 1256/250, T_2 = 1266/250,\\
&K = 3, r = 0.04.
\end{aligned}
$$

We examine the valuation of the spread option using the GSC model and GS model. The effect of linear relation, or cointegration under certain conditions, can be seen by comparing the GSC model with the GS model. Although the linear relation may include two or more commodity prices, here we assume that there are only two commodity prices in the linear relation.

Figure 1 illustrates the theoretical prices of commodity spread options on futures prices. We can see that the prices of the GSC model are lower than in the GS model. This is because the cointegrated prices tend to revert to satisfy the long-term relationship and hence do not diverge.

Sensitivities of commodity spread option prices to σ_{S_1} and σ_{S_2} are shown in Figure 2. The price calculated by the GSC model exhibits a u-shaped curve for both σ_{S_i}. Again, the prices of spread options obtained by the GSC model are lower than those in the GS model. This implies that the cointegration relation is in effect. That is, the long-term relationship force the commodity prices not to diverge.

[7] $h_1 = 1$ and $h_2 = 0.42$ are taken from the NYMEX crack spread between WTI crude oil and heating oil. Note that the payoff of this call spread option at maturity is $Max[h_2G_2 - h_1G_1 - K, 0]$.

Figure 1: Sensitivity of commodity spread option to future prices. The lower surface depicts the prices of commodity spread options obtained by the GSC model and the upper surface depicts the prices obtained by the GS model.

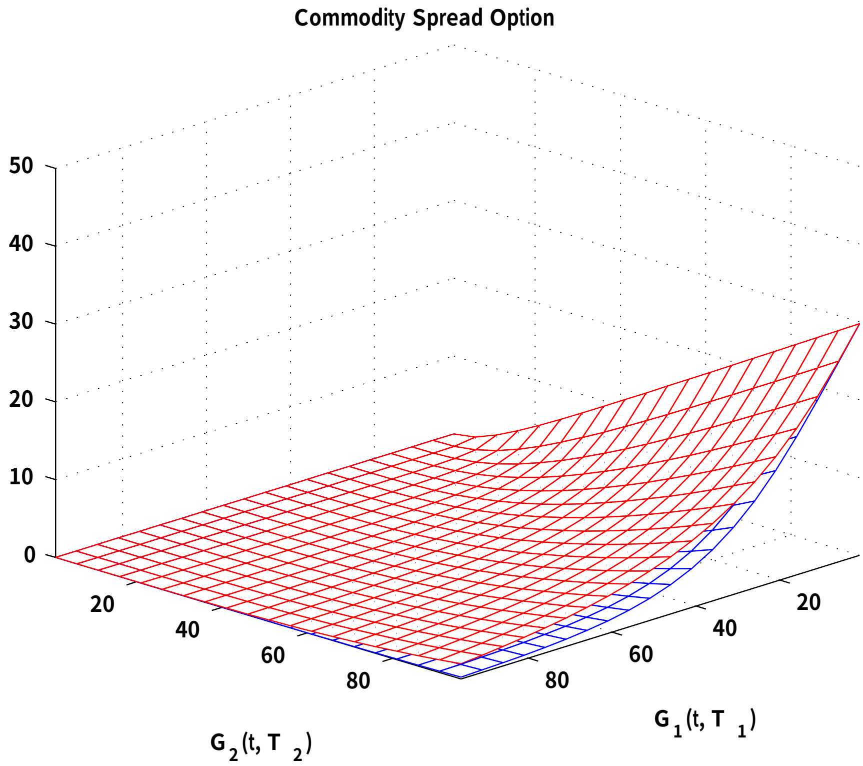
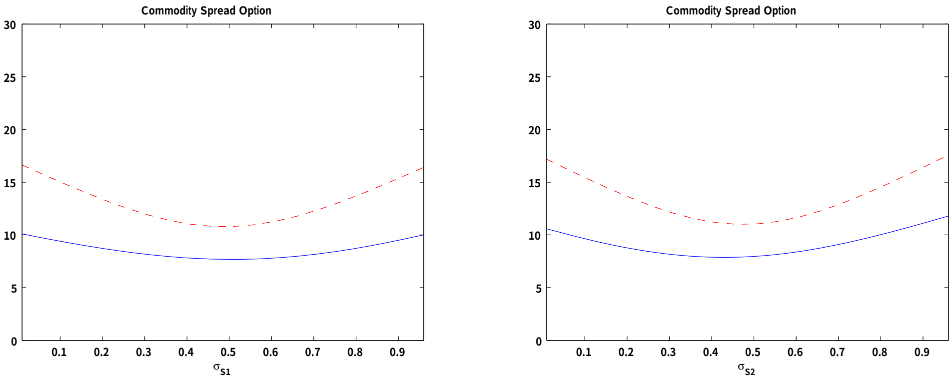


Figure 2: Sensitivity of commodity spread option to σ_{S_1} and σ_{S_2}. The solid line shows the prices of commodity spread options obtained by the GSC model and the dashed line shows the prices by the GS model.



In Figure 3, we report the sensitivity of commodity spread option prices with respect to b_1 and b_2. For b_1, the price exhibits a bell-shaped curve. There is a point where the price by the GSC model become larger than that of the GS model. This point is near the value where $b(= b_1 a_1 + b_2 a_2)$ become positive which means that the GSC model is not cointegrated.[8] Also b_1 have effect to the volatility of future price in a non-linear manner.[9] As $|b_2|$ increases (i.e., as b_2 decreases), the commodity spread option price decreases. The higher the absolute value of adjustment parameters is, the more quickly the spot price reverts to its long-term equilibrium level. In our setting, this reduces the spread and hence the price of spread option decreases. Again there is a point where the price by the GSC model become larger than that of the GS model which is occurs at the point where the value b become positive.

Figure 4 shows the sensitivity to κ_i. Here again, the prices calculated by the GSC model are lower than those of the GS model. The price in sensitivity analysis of κ_2 seems to converge to a certain level. For κ_1, the result imply that the prices of commodity spread option calculated by the GSC model may have already converged.

Finally, and most importantly Figure 5 shows the sensitivity of prices to

[8]For the GSC model to be cointegrated, we need $b < 0$. This is discussed in Nakajima and Ohashi (2012).

[9]See equation (8).

Figure 3: Sensitivity of commodity spread option to b_1 and b_2. The solid line shows the prices of commodity spread options obtained by the GSC model and the dashed line shows the prices by the GS model.

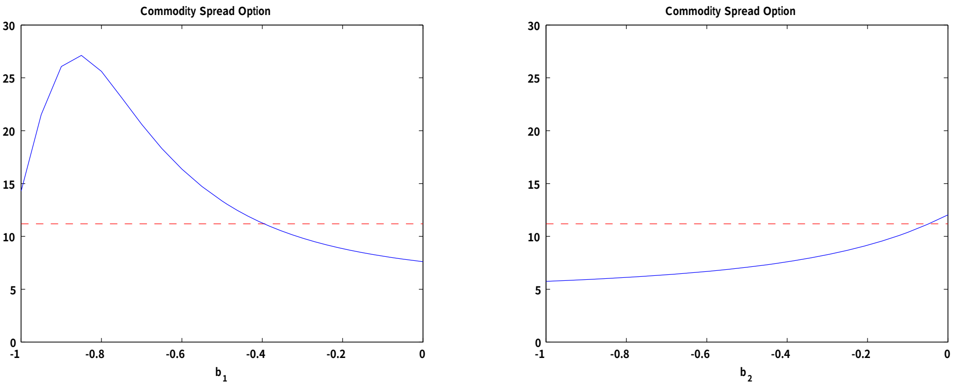
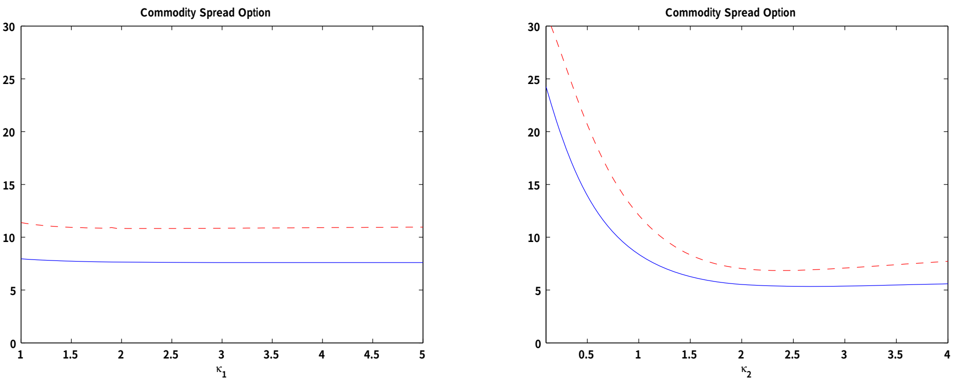


Figure 4: Sensitivity of commodity spread option to κ_1 and κ_2. The solid line shows the prices of commodity spread options obtained by the GSC model and the dashed line shows the prices by the GS model.



maturity. For GSC model, the option price have a inversed u-shaped curve. On the other hand, the option price by the GS model rises as maturity becomes longer. Notice that the longer the maturity is, the larger is the difference between the prices obtained by the GSC model and those of the GS model. The price obtained by the GSC model is about 1.5 time larger than those of the GS model when the maturity is around 6 years. This is because the cointegration relation prevents the commodity prices from diverging and hence makes the value of commodity spread options lower for longer maturities.

This result has imporant implication. That is, if cointegration exists, the GS model overprices the commodity spread option especially with longer maturities. Since the long-term commodity derivatives are often considered as useful tools to hedge risk against long-term projects such as large-scale oil refinery developments, it may be more appropriate to use the GSC model that incorporates cointegration rather than the GS model when pricing long-term derivatives.

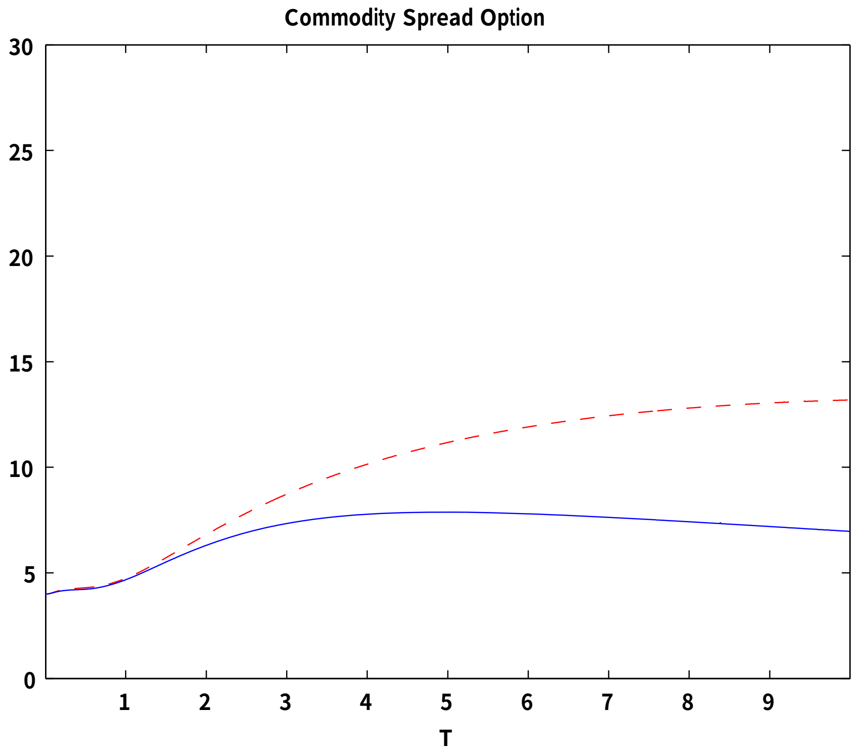
4 Conclusion

In this paper, we derive a valuation formula for European call commodity spread options and an analytical approximation formula of American call commodity spread options when commodity prices are cointegrated based on the GSC model developed by Nakajima and Ohashi (2012). We also derive the valuation formulae of commodity spread options for the GS model, which does not take account of cointegration, and compare the results for the GS and the GSC models.

With numerical analysis using the parameter values estimated in Nakajima and Ohashi (2012), we show that the prices of the commodity spread options given by the GSC model are lower than those of the GS model in most cases. This is because cointegrated commodity prices tend to revert to the long-term equilibrium level and hence do not diverge, which lowers the spread and hence the value of the spread option. The GSC model captures this phenomenon.

We also analyze the sensitivities of the commodity spread option price to the change of several parameter values. Among them, we find that as the maturity becomes longer, the difference between the price obtained by the GS model and that by the GSC model becomes larger, where the former is larger

Figure 5: Sensitivity of commodity spread option to maturity. The solid line shows the prices of commodity spread options obtained by the GSC model and the dashed line shows the prices by the GS model.



than the latter. This is again because the cointegration relation prevents the commodity prices from diverging. However, we also find that for shorter maturity, the spread of price options obtained by the GS model is smaller than that of the GSC model, which is consistent with Casassus, Liu, and Tang (2013). This implies that the GS model may overprice the commodity spread options for the longer maturity without taking account of cointegration. Since the long-term commodity derivatives are often considered as useful tools to hedge risk against long-term projects such as large-scale oil refinery developments, it may be more appropriate to use the GSC model that incorporates cointegration rather than the GS model when pricing long-term derivatives.

For future studies, further empirical analysis of derivative pricing that takes account of cointegration seems promising. In addition, it is interesting to see how cointegration affects prices of other types of derivatives such as a basket option, which is a generalization of a spread option. Incorporating other economic factors into the model, such as foreign exchange and/or interest rates also seems an important direction for future research.

5 Appendices

5.1 Expectation and Covariances of Log Futures Return

In this subsection, we derive the futures price and expectation value and covariance of log futures return. We use the future price equation written in terms of spot price and derive the future price process using Ito's lemma written in terms of futures price. This price process can be explicitly written in terms of futures price levels. Finally, we calculate the expectation and covariance of stochastic terms of futures price using properties of stochastic calculus.

Note that futures price in terms of spot price is[10]

$$G_i(t,T) = e^{\mu_{X_i}(t,T)+\frac{\sigma^2_{X_i}(t,T)}{2}},$$

where

$$\begin{aligned}
\beta_{S_i0}(t) &= r - \frac{\sigma^2_{S_i}}{2} + b_i\mu_z + b_i a_0 t,\\
\beta_{S_iS_j} &= b_i a_j,\\
\beta_{S_i\delta_i} &= -1,\\
\beta_{\delta_i0} &= \kappa_i\hat{\alpha}_i,\\
\beta_{\delta_i\delta_i} &= -\kappa_i,\\
\boldsymbol{\beta}_0(t) &= [\beta_{S_10}(t),\cdots,\beta_{S_n0}(t),\beta_{\delta_10},\cdots,\beta_{\delta_n0}]^\top,\\
\boldsymbol{\beta} &= \begin{bmatrix}
\beta_{S_1S_1} & \cdots & \beta_{S_1S_n} & \beta_{S_1\delta_1} & & 0\\
\vdots & \ddots & \vdots & & \ddots & \\
\beta_{S_nS_1} & \cdots & \beta_{S_nS_n} & 0 & & \beta_{S_n\delta_n}\\
 & & & \beta_{\delta_1\delta_1} & & 0\\
 & \mathbf{0} & & & \ddots & \\
 & & & 0 & & \beta_{\delta_n\delta_n}
\end{bmatrix},
\end{aligned}$$

[10]See Nakajima and Ohashi (2012) for derivation.

$$\begin{aligned}
\mu_{X_i}(t,T) &= E_t[\ln S_i(T)] \\
&= \left[e^{T\boldsymbol{\beta}} \left\{ e^{-t\boldsymbol{\beta}} \boldsymbol{X}(t) + \int_t^T e^{-s\boldsymbol{\beta}} \boldsymbol{\beta}_0(s) ds \right\} \right]_i, \\
\sigma_{X_i X_j}(t,T) &= E_t[(\ln S_i(T) - \mu_{X_i}(t,T))(\ln S_j(T) - \mu_{X_i}(t,T))] \\
&= \left[\int_t^T (e^{(T-s)\boldsymbol{\beta}}) \boldsymbol{\Sigma} (e^{(T-s)\boldsymbol{\beta}})^\top ds \right]_{ij}.
\end{aligned}$$

The partial derivatives are

$$\begin{aligned}
\frac{\partial G_i(t,T)}{\partial S_j(t)} &= \frac{\left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,j}}{S_j(t)} G_i(t,T), \\
\frac{\partial G_i(t,T)}{\partial \delta_j(t)} &= \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,n+j} G_j(t,T),
\end{aligned}$$

where we denote $[A]_{i,j}$ as $[i,j]$ th entry of matrix A.

Since the futures price $G_i(t,T)$ is a function of $S_i(t), \delta_i(t)$ and twice differentiable, we can use the Ito's lemma and the dynamics of future price is

$$dG_i(t,T) = \sum_{k=1}^n \sigma_{S_k} S_k(t) \frac{\partial G_i}{\partial S_k} dW_{S_k}(t) + \sum_{k=1}^n \sigma_{\delta_k} \frac{\partial G_i}{\partial \delta_k} dW_{\delta_k}(t),$$

where the drift term is 0 since $G_i(t,T)$ is martingale under the risk-neutral probability.

Again, using Ito's lemma we have,

$$\begin{aligned}
& d\log G_i(t,T) \\
= & -\frac{1}{2} \left\{ \sum_{k,l=1}^n \sigma_{S_k S_l} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,l} \right. \\
& + 2 \sum_{k,l=1}^n \sigma_{S_k \delta_l} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,n+l} \\
& \left. + \sum_{k,l=1}^n \sigma_{\delta_k \delta_l} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,n+k} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,n+l} \right\} dt \\
& + \sum_{k=1}^n \sigma_{S_k} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,k} dW_{S_k}(t) + \sum_{k=1}^n \sigma_{\delta_k} \left[e^{(T-t)\boldsymbol{\beta}} \right]_{i,n+k} dW_{\delta_k}(t).
\end{aligned}$$

The futures price can be expressed as follows.

$$G_i(T_0, T_i) = G_i(t, T_i) e^{X_{G_i}(t, T_0, T_i)}, \quad t \le T_0 \le T_i,$$

where

$$\begin{aligned} X_{G_i}(t, T_0, T_i) &\equiv \mu_{X_{G_i}}(t, T_0, T_i) \\ &+ \int_t^{T_0} \sum_{k=1}^{n} \sigma_{S_k} \left[e^{(T_i - t)\boldsymbol{\beta}} \right]_{i,k} dW_{S_k}(u) \\ &+ \int_t^{T_0} \sum_{k=1}^{n} \sigma_{\delta_k} \left[e^{(T_i - t)\boldsymbol{\beta}} \right]_{i,n+k} dW_{\delta_k}(u). \end{aligned}$$

The expectation value is

$$\begin{aligned} &\mu_{X_{G_i}}(t, T_0, T_i) \equiv E_t[X_{G_i}(t, T_0, T_i)] \\ = \ & -\frac{1}{2} \Bigg\{ \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{S_k S_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,l} du \\ &+ 2 \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{S_k \delta_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,n+l} du \\ &+ \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{\delta_k \delta_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,n+k} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,n+l} du \Bigg\}. \end{aligned} \tag{7}$$

The covariance of $\hat{X}_{G_i}(t, T_0, T_i)$ and $\hat{X}_{G_j}(t, T_0, T_j)$ is

$$\begin{aligned} &\sigma_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j) \equiv \mathrm{cov}_t[X_{G_i}(t, T_0, T_i), X_{G_i}(t, T_0, T_j)] \\ = \ & \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{S_k S_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T_j - u)\boldsymbol{\beta}} \right]_{j,l} du \\ &+ \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{S_k \delta_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,k} \left[e^{(T_j - u)\boldsymbol{\beta}} \right]_{j,n+l} du \\ &+ \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{S_l \delta_k} \left[e^{(T_j - u)\boldsymbol{\beta}} \right]_{j,l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,n+k} du \\ &+ \int_t^{T_0} \sum_{k,l=1}^{n} \sigma_{\delta_k \delta_l} \left[e^{(T_i - u)\boldsymbol{\beta}} \right]_{i,n+k} \left[e^{(T_j - u)\boldsymbol{\beta}} \right]_{j,n+l} du. \end{aligned} \tag{8}$$

5.2 Proof of Proposition 2.4

In this subsection, we prove Proposition 2.4. This is done by the following scheme. What we need to calculate is the expectation which can be expressed in terms of double integrals, since we are only dealing with the bivariate Gaussian processes. We know the expectation and covariance of the stochastic parts as we mentioned previously. The integrals can be calculated using multivariate version of completing squares, decomposing bivariate normal joint distribution in to conditional distribution and marginal distribution, and changing of variables. Finally, collecting all the terms, we have the pricing equation.

From Harrison and Kreps (1979) and Harrison and Pliska (1981), the price of commodity spread option at t, which option maturity is T_0, futures maturity for G_i and G_j are T_i and T_j, respectively, is

$$C^E(K, G_i, G_j, t, T_0, T_i, T_j) = e^{-r(T_0-t)} E_t[(h_i G_i(T_0, T_i) - h_j G_j(T_0, T_j) - K)^+].$$

The expectation value can be calculated as follows.

$$\begin{aligned} & E_t[(h_i G_i(T_0, T_i) - h_j G_j(T_0, T_j) - K)^+] \\ = & \int_D (h_i G_i(t, T_i) e^{x_i} - h_j G_(t, T_j) e^{x_j} - K) n(\boldsymbol{x} | \boldsymbol{\mu_x}, \boldsymbol{\Sigma_x}) d\boldsymbol{x}, \end{aligned}$$

where

$$\begin{aligned} \boldsymbol{x} &= \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, \\ \boldsymbol{\mu_x} &= \begin{bmatrix} \mu_{X_{G_i}}(t, T_0, T_i) \\ \mu_{X_{G_j}}(t, T_0, T_j) \end{bmatrix}, \\ \boldsymbol{\Sigma_x} &= \begin{bmatrix} \sigma^2_{X_{G_i}}(t, T_0, T_i, T_i) & \sigma_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j) \\ \sigma_{X_{G_i} X_{G_j}}(t, T_0, T_i, T_j) & \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j) \end{bmatrix}, \end{aligned}$$

and

$$\begin{aligned} D &= \{\boldsymbol{x} | h_i G_i(t, T_i) e^{x_i} - h_j G_j(t, T_j) e^{x_j} - K \geq 0\} \\ &= \{\boldsymbol{x} | \ln(h_j G_j(t, T_j) e^{x_j} + K) - \ln(h_i G_i(t, T_i)) \leq x_i\} \\ &= \{\boldsymbol{x} | d(x_j) \leq x_i\}, \\ d(x_j, K) &\equiv \ln(h_j G_j(t, T_j) e^{x_j} + K) - \ln(h_i G_i(t, T_i)). \end{aligned}$$

We now calculate the integrals. Suppose that $\boldsymbol{e}_i$ is unit vector which the ith row is 1.

$$
\begin{aligned}
& \int_D e^{x_i} n(\boldsymbol{x}|\boldsymbol{\mu_x}, \boldsymbol{\Sigma_x}) d\boldsymbol{x} \\
= & \int_D (2\pi)^{-1} |\boldsymbol{\Sigma_x}|^{-\frac{1}{2}} \exp\{\boldsymbol{e}_i^\top \boldsymbol{x} - \frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_x})^\top \boldsymbol{\Sigma_x}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_x})\} d\boldsymbol{x} \\
= & \int_D (2\pi)^{-1} |\boldsymbol{\Sigma_x}|^{-\frac{1}{2}} \exp\{\boldsymbol{e}_i^\top \boldsymbol{\mu_x} + \boldsymbol{e}_i^\top (\boldsymbol{x} - \boldsymbol{\mu_x}) - \frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_x})^\top \boldsymbol{\Sigma_x}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_x})\} d\boldsymbol{x} \\
= & \int_D (2\pi)^{-1} |\boldsymbol{\Sigma_x}|^{-\frac{1}{2}} \exp\Bigg\{ \boldsymbol{e}_i^\top \boldsymbol{\mu_x} + \frac{1}{2} \boldsymbol{e}_i^\top \boldsymbol{\Sigma_x} \boldsymbol{e}_i \\
& - \frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x} \boldsymbol{e}_i)^\top \boldsymbol{\Sigma_x}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x} \boldsymbol{e}_i) \Bigg\} d\boldsymbol{x} \\
= & \exp\Bigg\{ \mu_{X_{G_i}}(t, T_0, T_i) + \frac{1}{2} \sigma^2_{X_{G_i}}(t, T_0, T_i, T_i) \Bigg\} \\
& \times \int_D (2\pi)^{-1} |\boldsymbol{\Sigma_x}|^{-\frac{1}{2}} \exp\{ -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x} \boldsymbol{e}_i)^\top \boldsymbol{\Sigma_x}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x} \boldsymbol{e}_i)\} d\boldsymbol{x}.
\end{aligned}
$$

The integral can be expanded as follows. We omit the time prameters for simplicity.

$$
\begin{aligned}
& \int_D (2\pi)^{-1} |\boldsymbol{\Sigma_x}|^{-\frac{1}{2}} \exp\Bigg\{ -\frac{1}{2}(\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x} \boldsymbol{e}_i)^\top \boldsymbol{\Sigma_x}^{-1} (\boldsymbol{x} - \boldsymbol{\mu_x} - \boldsymbol{\Sigma_x} \boldsymbol{e}_i) \Bigg\} d\boldsymbol{x} \\
= & \int_{-\infty}^{\infty} \int_{d(x_j, K)}^{\infty} (2\pi)^{-1} (\sigma_{X_{G_i}} \sigma_{X_{G_j}} \sqrt{1 - \rho^2_{X_{G_i} X_{G_j}}})^{-1} \exp\Bigg\{ -\frac{1}{2(1 - \rho_{X_{G_i} X_{G_j}})} \\
& \times \Bigg(\left(\frac{x_i - \mu_{X_{G_i}} - \sigma^2_{X_{G_i}})}{\sigma_{X_{G_i}}} \right)^2 - 2\rho_{X_{G_i} X_{G_j}} \left(\frac{x_i - \mu_{X_{G_i}} - \sigma^2_{X_{G_i}}}{\sigma_{X_{G_i}}} \right) \\
& \times \left(\frac{x_j - \mu_{X_{G_j}} - \sigma_{X_{G_i} X_{G_j}}}{\sigma_{X_{G_j}}} \right) + \left(\frac{x_j - \mu_{X_{G_j}} - \sigma_{X_{G_i} X_{G_j}}}{\sigma_{X_{G_j}}} \right)^2 \Bigg)^2 \Bigg\} dx_i dx_j
\end{aligned}
$$

$$
\begin{aligned}
= & \int_{-\infty}^{\infty}\int_{d(x_j,K)}^{\infty}(2\pi(1-\rho^2_{X_{G_i}X_{G_j}}))^{-\frac{1}{2}}\sigma^{-1}_{X_{G_i}} \\
& \times\exp\left\{-\frac{\left(x_i-\mu_{X_{G_i}}-\sigma^2_{X_{G_i}}-\rho_{X_{G_i}X_{G_j}}\sigma_{X_{G_i}}\frac{x_j-\mu_{X_{G_j}}-\sigma_{X_{G_i}X_{G_j}}}{\sigma_{X_{G_j}}}\right)^2}{2(1-\rho^2_{X_{G_i}X_{G_j}})\sigma^2_{X_{G_i}}}\right\}dx_i \\
& \times(2\pi)^{-\frac{1}{2}}\sigma^{-1}_{X_{G_j}}\exp\left\{-\frac{1}{2}\left(\frac{x_j-\mu_{X_{G_j}}-\sigma_{X_{G_i}X_{G_j}}}{\sigma_{X_{G_j}}}\right)^2\right\}dx_j \\
= & \int_{-\infty}^{\infty}\int_{-d_i(x_j,K)}^{\infty}(2\pi(1-\rho^2_{X_{G_i}X_{G_j}}))^{-\frac{1}{2}}\sigma^{-1}_{X_{G_i}} \\
& \times\exp\left\{-\frac{y^2}{2}\right\}(1-\rho^2_{X_{G_i}X_{G_j}})^{\frac{1}{2}}\sigma_{X_{G_i}}dy \\
& \times(2\pi)^{-\frac{1}{2}}\sigma^{-1}_{X_{G_j}}\exp\left\{-\frac{1}{2}\left(\frac{x_j-\mu_{X_{G_j}}-\sigma_{X_{G_i}X_{G_j}}}{\sigma_{X_{G_j}}}\right)^2\right\}dx_j \\
= & \int_{-\infty}^{\infty}\Phi(d_i(x_j,K))n(x_j|\mu_{X_{G_j}}+\sigma_{X_{G_i}X_{G_j}},\sigma^2_{X_{G_j}})dx_j,
\end{aligned}
$$

where

$$
d_i(x_j,K)=-\frac{d(x_i,K)-\mu_{X_{G_i}}-\sigma^2_{X_{G_i}}-\rho_{X_{G_i}X_{G_j}}\sigma_{X_{G_i}}\frac{x_j-\mu_{X_{G_j}}-\sigma_{X_{G_i}X_{G_j}}}{\sigma_{X_{G_j}}}}{\sigma_{X_{G_i}}\sqrt{(1-\rho^2_{X_{G_i}X_{G_j}})}},
$$

and we used change of variables in the third equation. Other integrals can be derived in the same manner. For the second integral,

$$
\begin{aligned}
& \int_D(2\pi)^{-1}|\boldsymbol{\Sigma}_{\boldsymbol{x}}|^{-\frac{1}{2}}\exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{x}}-\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{e}_j)^\top\boldsymbol{\Sigma}^{-1}_{\boldsymbol{x}}(\boldsymbol{x}-\boldsymbol{\mu}_{\boldsymbol{x}}-\boldsymbol{\Sigma}_{\boldsymbol{x}}\boldsymbol{e}_j)\right\}d\boldsymbol{x} \\
= & \int_{-\infty}^{\infty}\Phi(d_j(x_j,K))n(x_j|\mu_{X_{G_j}}+\sigma^2_{X_{G_j}},\sigma^2_{X_{G_j}})dx_j,
\end{aligned}
$$

where

$$d_j(x_j, K) = -\frac{d(x_i, K) - \mu_{X_{G_i}} - \sigma_{X_{G_i}X_{G_j}} - \rho_{X_{G_i}X_{G_j}}\sigma_{X_{G_i}}\frac{x_j - \mu_{X_{G_j}} - \sigma^2_{X_{G_j}}}{\sigma_{X_{G_j}}}}{\sigma_{X_{G_i}}\sqrt{(1-\rho^2_{X_{G_i}X_{G_j}})}}.$$

And the last integral is

$$\begin{aligned} & \int_D (2\pi)^{-1}|\boldsymbol{\Sigma_x}|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}(\boldsymbol{x}-\boldsymbol{\mu_x})^\top \boldsymbol{\Sigma_x}^{-1}(\boldsymbol{x}-\boldsymbol{\mu_x})\right\} d\boldsymbol{x} \\ = & \int_{-\infty}^{\infty} \Phi(d_j(x_j, K)) n(x_j|\mu_{X_{G_j}}, \sigma^2_{X_{G_j}}) dx_j, \end{aligned}$$

where

$$d_i(x_j, K) = -\frac{d(x_i, K) - \mu_{X_{G_i}} - \rho_{X_{G_i}X_{G_j}}\sigma_{X_{G_i}}\frac{x_j - \mu_{X_{G_j}}}{\sigma_{X_{G_j}}}}{\sigma_{X_{G_i}}\sqrt{(1-\rho^2_{X_{G_i}X_{G_j}})}}.$$

Collecting all terms, we have

$$\begin{aligned} & C^E(G_i, G_j, t, T_0, T_i, T_j) \\ = & h_i G_i(t, T_i) \exp\{-r(T_0 - t)\} \\ & \times \int_{-\infty}^{\infty} \Phi(d_i(x_j, K)) n(x_j|\mu_{X_{G_j}}(t, T_0, T_j) + \sigma_{X_{G_i}X_{G_j}}(t, T_0, T_i, T_j), \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j)) dx_j \\ & - h_j G_j(t, T_j) \exp\{-r(T_0 - t)\} \\ & \times \int_{-\infty}^{\infty} \Phi(d(x_j, K)) n(x_j|\mu_{X_{G_j}}(t, T_0, T_j) + \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j), \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j)) dx_j \\ & - K e^{-r(T_0 - t)} \int_{-\infty}^{\infty} \Phi(d(x_j, K)) n(x_j|\mu_{X_{G_j}}(t, T_0, T_j), \sigma^2_{X_{G_j}}(t, T_0, T_j, T_j)) dx_j, \end{aligned}$$

where

$$\begin{aligned} d(x_j, K) = & -\frac{\ln(h_j G_j(t, T_j) e^{x_j} + K) - \ln(h_i G_i(t, T_i)) - \mu_{X_{G_i}}(t, t_0, T_i)}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i)\sqrt{1 - \rho^2_{X_{G_i}X_{G_j}}(t, T_0, T_i, T_j)}} \\ & + \frac{\rho_{X_{G_i}X_{G_j}}(t, T_0, T_i, T_j)\sigma_{X_{G_i}}(t, T_0, T_i, T_i)\frac{x_j - \mu_{X_{G_j}}(t, T_0, T_j)}{\sigma_{X_{G_j}}(t, T_0, T_i, T_j)}}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i)\sqrt{1 - \rho^2_{X_{G_i}X_{G_j}}(t, T_0, T_i, T_j)}}, \end{aligned}$$

$$d_i(x_j, K) = d(x_j, K) + \frac{\sigma^2_{X_{G_i}}(t, T_0, T_i, T_i) - \rho^2_{X_{G_i}X_{G_j}}(t, T_0, T_i, T_j)\sigma^2_{X_{G_i}}(t, T_0, T_i, T_i)}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i)\sqrt{1 - \rho^2_{X_{G_i}X_{G_j}}(t, T_0, T_i, T_j)}},$$

$$\rho_{X_{G_i}X_{G_j}}(t, T_0, T_i, T_j) = \frac{\sigma_{X_{G_i}X_{G_j}}(t, T_0, T_i, T_j)}{\sigma_{X_{G_i}}(t, T_0, T_i, T_i)\sigma_{X_{G_j}}(t, T_0, T_j, T_j)}.$$

5.3 Analytical Approximation for American Commodity Spread Option

In this subsection, we propose an analytical approximation pricing formula for American commodity spread option. The difficulty is the calculation of early exercise premium, more precisely the domain of integration or the condition inequality of exercise which are not analytically tractable. Therefore, we use the scheme of Bjerksund and Stensland (1994) to approximate the condition inequality which split spread option in to two call option that the first option has stochastic exercise price and then use Barone-Adesi and Whaley (1987) framework to approximate the two American option. The formula can now be calculated as we did in European call option which derives the analytical approximated pricing formula for American commodity spread options.

From Broadie and Detemple (1997) the valuation of American spread options are[11]

$$
\begin{aligned}
& C^A(K, G_i, G_j, t, T_0, T_i, T_j) \\
= & \sup_{\tau \in \mathcal{S}_{t,T_0}} E_t[e^{-r(\tau-t)}(h_i G_i(\tau, T_i) - h_j G_j(\tau, T_j) - K)^+] \\
= & C^E(K, G_i, G_j, t, T_0, T_i, T_j) + a^A(K, G_i, G_j, t, T_0, T_i, T_j; B^{c,s}),
\end{aligned}
$$

where $\mathcal{S}_{t,T_0}$ is the class of stopping times of the filtration generated by the underlying the Brownian motion processes, the early exercise premium a^A is defined by

$$
\begin{aligned}
& a^A(K, G_i, G_j, t, T_0, T_i, T_j; B^{c,s}) \\
= & E_t\left[\int_t^{T_0} e^{-r(u-t)}(rh_i G_i(u, T_i) - rh_j G_j(u, T_j) - rK) \right. \\
& \left. \times 1_{\{h_i G_i(u,T_i) \geq B^{c,s}(G_j(u,T_j),u)\}} du \right] \\
= & r \int_t^{T_0} e^{-r(u-t)} E_t[(h_i G_i(u, T_i) - h_j G_j(u, T_j) - K) \\
& \times 1_{\{h_i G_i(u,T_i) \geq B^{c,s}(G_j(u,T_j),u)\}}] du,
\end{aligned}
$$

[11]See also Detemple (2006), Section 6.4. In this section, we have omitted some time parameters.

and $B^{c,s}(\cdot,\cdot)$ is a solution to the integral equation

$$B^{c,s}(G_j(t,T_j),t) - K = C^E(K, G_i, B^{c,s}, t, T_0, T_i, T_j) + a^A(K, G_i, B^{c,s}, t, T_0, T_i, T_j; B^{c,s}),$$

subject to

$$\begin{aligned} \lim_{t\to T} B^{c,s}(G_j(t,T_j),t) &= \max(G_i(t,T_i)+K, G_j(t,T_j)+K), \\ B^{c,s}(0,t) &= B^j(t), \\ B^j(t) &= \inf\{G_j(t,T_j) : C^A(K, G_i, G_j, t, T_0, T_i, T_j) \\ &\qquad = (h_i G_i(t,T_i) - h_j G_j(t,T_j) - K)^+\}. \end{aligned}$$

Now, we addopt the framework of Bjerksund and Stensland (1994)[12] to approximate the early exercise premium.

$$\begin{aligned} & E_t[(h_i G_i(u,T_i) - h_j G_j(u,T_j) - K)1_{\{h_i G_i(u,T_i) \geq B^{c,s}(G_j(u,T_j),u)\}}] \\ \approx\ & E_t[(h_i G_i(u,T_i) - h_j G_j(u,T_j) - K)1_{\{h_i G_i(u,T_i) \geq h_j B_j G_j(u,T_j) + B_K K\}}], \end{aligned}$$

where

$$\begin{aligned} B_j &= B(T_0 - u, \sigma^2_{X_{G_i}}(u,T_0) - 2\sigma_{X_{G_i}X_{G_j}}(u,T_0) + \sigma^2_{X_{G_j}}(u,T_0)), \\ B_K &= B(T_0 - u, \sigma^2_{X_{G_i}}(u,T_0)), \\ B(t,\sigma^2) &= e^{h(t,\sigma^2)} + (1 - e^{h(t,\sigma^2)})B_\infty(\sigma^2), \\ B_\infty(\sigma^2) &= \frac{\beta(\sigma^2)}{\beta(\sigma^2)-1}, \\ \beta(\sigma^2) &= \frac{1}{2} + \sqrt{\frac{1}{4} + \frac{2r}{\sigma^2}}, \\ h(t,\sigma^2) &= -2\sigma\sqrt{t}(\beta(\sigma^2)-1). \end{aligned}$$

The approximation is constructed in two steps. The first step is to split the spread option into an exchange option and vanilla type option. And the second step is due to Barone-Adesi and Whaley (1987) framework of approximating American option.

Note that the exercise region is

$$\begin{aligned} & h_i G_i(u,T_i) \geq h_j B_j G_j(u,T_j) + B_K K \\ & \Leftrightarrow x_i \leq d_{eep0}(x_j,u,K) \equiv \ln(h_j B_j G_j(t,T_j)e^{x_j} + B_K K) - \ln(h_i G_i(t,T_i)). \end{aligned}$$

[12] They have also analyzed the performance of their approach.

The integrals of early exercise premium can be calculated just as the integral of European commodity spread option which we now have

$$
\begin{aligned}
& a^A(K, G_i, G_j, t, T_0, T_i, T_j; B^{c,s}) \\
= \; & r\int_t^{T_0} e^{-r(u-t)} E_t[(h_i G_i(u,T_i) - h_j G_j(u,T_j) - K) \\
& \times 1_{\{h_i G_i(u,T_i) \geq B^{c,s}(G_j(u,T_j),u)\}}] du \\
\approx \; & r\int_t^{T_0} e^{-r(u-t)} E_t[(h_i G_i(u,T_i) - h_j G_j(u,T_j) - K) \\
& \times 1_{\{x_i \leq d_{eep0}(x_j,u,K)\}}] du \\
= \; & r\Bigg[h_i G_i(t,T_i) \int_t^{T_0} \exp\left\{ -r(u-t) + \mu_{X_{G_i}}(t,u,T_i) + \frac{1}{2}\sigma^2_{X_{G_i}}(t,u,T_i,T_i) \right\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_{eep2}(x_j,u,K)) \\
& \times n(x_j | \mu_{X_{G_j}}(t,u,T_j) + \sigma_{X_{G_i}X_{G_j}}(t,u,T_i,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \\
& - h_j G_j(t,T_j) \int_t^{T_0} \exp\left\{ -r(u-t) + \mu_{X_{G_j}}(t,u,T_j) + \frac{1}{2}\sigma^2_{X_{G_j}}(t,u,T_j,T_j) \right\} \\
& \times \int_{-\infty}^{\infty} \Phi(d_{eep1}(x_j,u,K)) \\
& \times n(x_j | \mu_{X_{G_j}}(t,u,T_j) + \sigma^2_{X_{G_j}}(t,u,T_j,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \\
& - K \int_t^{T_0} e^{-r(u-t)} \int_{-\infty}^{\infty} \Phi(d_{eep1}(x_j,u,K)) \\
& \times n(x_j | \mu_{X_{G_j}}(t,u,T_j), \sigma^2_{X_{G_j}}(t,u,T_j,T_j)) dx_j du \Bigg],
\end{aligned}
$$

where

$$
\begin{aligned}
d_{eep1}(x_j,u,K) \;=\; & -\frac{\ln(h_j B_j G_j(t,T_j)e^{x_j} + B_K K) - \ln(h_i G_i(t,T_i)) - \mu_{X_{G_i}}(t,u,T_i)}{\sigma_{X_{G_i}}(t,u,T_i)\sqrt{1-\rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)}} \\
& + \frac{\rho_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)\sigma_{X_{G_i}}(t,u,T_i)\frac{x_j - \mu_{X_{G_j}}(t,u,T_j)}{\sigma_{X_{G_j}}(t,u,T_j,T_j)}}{\sigma_{X_{G_i}}(t,u,T_i)\sqrt{1-\rho^2_{X_{G_i}X_{G_j}}(t,u,T_i,T_j)}},
\end{aligned}
$$

$$d_{eep2}(x_j, u, K) = d_{eep}(x_j, u, K) + \frac{\sigma^2_{X_{G_i}}(t, u, T_i, T_i) - \rho^2_{X_{G_i}X_{G_j}}(t, u, T_i, T_j)\sigma^2_{X_{G_i}}(t, u, T_i, T_i)}{\sigma_{X_{G_i}}(t, u, T_i)\sqrt{1 - \rho^2_{X_{G_i}X_{G_j}}(t, u, T_i, T_j)}}.$$

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