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Random thinning with credit quality vulnerability factor for better risk management of credit portfolio in a top-down framework [1]

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Abstract

In the top-down approach of portfolio credit risk modeling, we assess credit risks of sub-portfolios with the so-called random thinning model, which dissects the portfolio risk into sub-portfolio contributions. In this paper, we provide a random thinning model incorporating the sub-portfolio size and the factor called "credit quality vulnerability factor", in order to take into account credit quality vulnerability of sub-portfolios. With our random thinning model, we estimate credit quality vulnerability of industrial sectors. Numerical examples on assessing the risks of several credit portfolios show that our random thinning model is useful to detect how the proportions of constituent industrial sectors affect portfolio credit risks.

1 Introduction

Quantitative credit risk model is essential for financial institutions to quantify and control their credit exposure. In the literature of modeling credit risk for portfolios, there are two different paradigms of "bottom-up" and "top-down".

In bottom-up approaches, we start to model credit risk of each obligor who is a constituent of some credit portfolio, and then we aggregate all the

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risks of obligors in the portfolio to assess the portfolio-level credit risk in consideration of dependence structure among the portfolio constituents (for example, see Duffie and Singleton [2], Lando [10]).

On the other hand, often in top-down approach, an intensity-based model of the total credit loss point process in the underlying portfolio is primarily specified without reference to the portfolio constituents, and (if necessary) the procedure called random thinning is used to obtain individual intensities of the sub-portfolio or the portfolio constituents.

The purpose of this paper is to construct a new random thinning model in the top-down framework so as to precisely assess credit event risk of sub-portfolios in the economy-wide portfolio.

Credit risk modeling within the top-down approach, suggested by Giesecke et al.[4], starts with constructing portfolio-level (or economy-level) loss point processes, rather than loss point processes for the portfolio constituent obligors. Since Giesecke et al.[4], the literature on top-down approach credit risk modeling has expanded: Errais et al.[3], Kunisch and Uhrig-Homburg [9], Halperin and Tomecek [7], Nakagawa [12, 13], Kaneko and Nakagawa [8], Giesecke and Kim [5, 6], Yamanaka et al.[15, 16].

As Halperin and Tomecek [7] pointed out, a portfolio-level (or economy-wide-level) loss process of the top-down approach is generally much easier to calibrate to prices of credit portfolio derivatives than most aggregate portfolio loss processes obtained with the bottom-up approach. In addition to this, credit contagion can be readily introduced in this approach (for instance see Giesecke et al.[4], Giesecke and Kim [5] and Yamanaka et al.[15, 16]).

For the purpose of evaluating credit risk of the sub-portfolios or the constituent obligors in the top-down approach, usually used is the idea of random thinning, that is, allocating the whole credit event intensity for the "top-part" portfolio among the sub-portfolios or the constituents that decompose the whole portfolio.

Various versions of the random thinning model are proposed in previous works as follows. Giesecke et al.[4] and Halperin and Tomecek [7] proposed piece-wise constant thinning model for calculating portfolio constituent firms risk contribution and derived pricing formula for credit default swap (CDS) with the model. Kunisch and Uhrig-Homburg [9] and Kaneko and Nakagawa [8] proposed thinning models in which the reference constituent default probability is obtained by Merton model (see Merton [11]); Kunisch and Uhrig-Homburg [9] used the thinning model to derive CDS pricing formula, while Kaneko and Nakagawa [8] did to study pricing of lending interest rate for bank loan. Giesecke and Kim [6] used a thinning model to obtain default intensity of the financial sector in US.

Giesecke and Kim [5] dissected the portfolio risk into sub-portfolio contributions by the thinning process specified by the weight of the sub-portfolio constituents to the whole portfolio. With such a thinning process, they analyzed the risk of collateralized debt obligations (CDOs) and suggested the method of generating the credit event time samples of the target sub-portfolios. Also they obtained default intensity of financial sectors with thinning models which are driven by macroeconomic factors.

Yamanaka et al. [15, 16], similar to Giesecke and Kim [5], used the thinning process specified by the weight of the sub-portfolio to the whole portfolio to calculate several risk measures such as Value at Risk(VaR) and Expected Shortfall(ES) of sub-portfolios.

Although there have been several studies on random thinning as above, it appears insufficient to evaluate the credit risk of the sub-portfolios or individual obligors consistently with the top-part risk evaluation. Indeed it seems that the credit event frequency in each subset in the economy cannot be explained only by the weight of the sub-portfolio constituents to the whole portfolio.

Therefore, for more precise credit risk assessment of the target sub-portfolios in the top-down framework, we try introducing a new thinning model which is a kind of generalization of the thinning model proposed in Giesecke and Kim [5] and Yamanaka et al. [15, 16]. Specifically the thinning model is the product of a couple of components: one is the same as the previous works, that is, the ratio of the number of constituent obligors in the sub-portfolio to that of the whole portfolio, and the other is another factor regarded as the vulnerability measure of credit quality of each sub-portfolio. We call the factor "credit quality vulnerability factor (CQVF)" in this paper. Here, the term "vulnerability" means how the credit qualities of the portfolio constituents tend to change. Especially, we suppose a constant CQVF ,for simplicity, associated with each of some sub-portfolios that partition the economy-wide portfolio according to some similarity in industrial classification. With such a thinning model, we examine the peculiarity of such partitioning sub-portfolios through estimation of the constant CQVF, so that we can confirm that it is useful to introduce CQVF to the thinning model even though it is the simple constant case. Moreover, as a numerical example, we simulate the future frequency of credit events for several portfolios that have different industrial distributions, in other words, different vulnerability of credit quality.

This paper is organized as follows. Section 2 summarizes the top-down approach. We propose some thinning models incorporating CQVF in section 3. In section 4, we suppose that the economy-wide portfolio is partitioned into five sub-portfolios. Under the assumption, we present the maximum likelihood framework for estimating constant CQVFs, then estimate the constant CQVFs for the partitioning sub-portfolios with the historical data on rating transitions of Japanese enterprises, and apply the constant CQVF model to risk analysis of some hypothetical portfolios at last. We presents some concluding remarks in section 5 and we discuss some complementary topics in appendices.

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2 Framework of top-down approach

In this section, we present a framework of top-down approach for credit risk modeling. We model uncertainty in the economy on a filtered complete probability space $(\Omega,\ \mathcal{F},\ \mathbb{P},\ \{\mathcal{F}_t\}_{t\geq 0})$, where $\{\mathcal{F}_t\}_{t\geq 0}$ is a right-continuous and complete filtration.

We implicitly suppose that there are countable obligors which can be faced with some credit events[2] such as defaults as well as credit rating transitions in our credit portfolio. We denote the set of all of such obligors by $S^* = \{s_1, s_2, \ldots\}$.

Let $\mathcal{T}^* = \{T_n\}_{n=1,2,\cdots}$ be a strictly increasing sequence, with $0 < T_1 < T_2 < \cdots$, of totally inaccessible $\{\mathcal{F}_t\}$-stopping times. We regard the sequence of stopping times $\{T_n\}_{n=1,2,\cdots}$ as the ordered credit event times observed in the economy-wide portfolio S^*.

Moreover, we specify by $N_t^* = \sum_{n\geq 1} \mathbf{1}_{\{T_n \leq t\}}$ the counting process of credit events in economy-wide S^*, and suppose that N_t^* has an intensity process λ_t^*, namely , λ_t^* is a $\{\mathcal{F}_t\}$-progressively measurable non-negative process such that the compensated process $N_t^* - \int_0^t \lambda_s^* \mathrm{d}s$ is an $\{\mathcal{F}_t\}$-local martingale. We call λ_t^* the total intensity (process).

When we need to focus on credit events happened in some sub-portfolio $S(\subset S^*)$, it is necessary to have the intensity process specified for the sub-portfolio S, denoted by λ_t^S. In order to obtain such intensity, we have to mention the procedure, which is often called a random thinning, of partitioning the total intensity λ_t into the intensity λ_t^S for sub-portfolio S.

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[2] Credit events considered hereafter are default as well as credit rating transition (downgrade and/or upgrade). Giesecke et al.[4], Errais et al.[3], Kunisch and Uhrig-Homburg [9], Halperin and Tomecek [7], Giesecke and Kim [5] focused on defaults. Nakagawa [12] and Yamanaka et al.[15] focused on credit rating transitions. Kaneko and Nakagawa [8] and Yamanaka et al.[16] considered both rating transitions and defaults.

Assume that the set S^* is decomposed into $M(\geq 2)$ non-empty sub-portfolios $\{S_i\}_{i=1,\cdots,M}$ so that $S_1 \cup \cdots \cup S_M = S^*$ and $S_i \cap S_j = \emptyset$ $(i \neq j)$. The random thinning is specified by an $\{\mathcal{F}_t\}$-adapted processes $\{Z_t^i\}_{i=1,\cdots,M}$ with $0 \leq Z_t^i \leq 1$ for any $i = 1, \cdots, M$, which satisfies the following condition:

$$\sum_{i=1}^{M} Z_t^i = 1 \text{ a.s. for any } t \geq 0. \tag{1}$$

Hereafter we use simple super-indices like writing Z_t^i for $Z_t^{S_i}$, if the sub-portfolios are labeled by integers and not confusing.

For any sub-portfolio S, the variable Z_t^S can be seen as the conditional probability that, when a credit event occurs at time t in S^*, the event is observed in sub-portfolio S. The process $\{Z_t^S\}$ is called a thinning process associated with sub-portfolio S.

We remark that the counting process N_t^S of credit events observed in sub-portfolio S is specified by

$$N_t^S = \sum_{n \geq 1} \mathbf{1}_{\{T_n \leq t\} \cap \{T_n \in \mathcal{T}^S\}}, \tag{2}$$

where $\mathcal{T}^S$ stands for the set of event times observed in sub-portfolio S.

From Proposition 2.1 in Giesecke and Kim [5], the intensity associated with the counting process N_t^S is given by the process λ_t^S defined as follows:

$$\lambda_t^S = Z_t^S \lambda_t^*, \tag{3}$$

where λ_t^* is the total intensity and Z_t^S is the thinning process.

Thus the expected number given the information up to time t of credit events happened in sub-portfolio S during the future period $(s, u]$ $(t \leq s < u)$ can be obtained by

$$\mathbf{E}\left[N_u^S - N_s^S \mid \mathcal{F}_t\right] = \int_s^u \mathbf{E}\left[\lambda_v^S \mid \mathcal{F}_t\right] dv = \int_s^u \mathbf{E}\left[Z_v^S \lambda_v^* \mid \mathcal{F}_t\right] dv, \tag{4}$$

where $\mathbf{E}[\cdot]$ stands for the expectation under the probability measure P.

3 Thinning model with credit quality vulnerability factor

This section introduces a thinning model for assessing credit risks of sub-portfolios. The model is specified by portfolio size (the number of constituent obligors in the sub-portfolio) as well as another factor which represents a credit quality vulnerability of the sub-portfolio.

We simply consider that the frequency of credit event occurrences depend on the size of the portfolio. Thus, we introduce the ratio of the number of constituents in sub-portfolio S to that of the economy-wide portfolio S^* as follows:

$$\tilde{Z}_t^S = \frac{X_t^S}{X_t^*}, \tag{5}$$

where X_t^S (resp. X_t^*) denotes the number of obligors contained at time t in portfolio S (resp. economy-wide portfolio S^*). The quotients (5) is taken to be 0 when the denominator vanishes.

Although $\tilde{Z}_t^S$ itself seems a natural candidate of the thinning process for S, we rather specify a thinning process Z_t^S by

$$Z_t^S = \theta_t^S \tilde{Z}_t^S, \tag{6}$$

where $\{\theta_t^S\}_{t\geq 0}$ is a positive $\{\mathcal{F}_t\}$-adapted process[3].

We can see θ_t^S as some factor that affects credit event frequency independent of the portfolio size effect. If θ_t^S becomes larger, the credit event frequency of sub-portfolio S higher and thus the credit quality of the sub-portfolio would be more vulnerable. On the other hand, if θ_t^S smaller, the credit events in portfolio S happen less and thus the credit quality of S would be less vulnerable. Hence we call θ_t^S "Credit Quality Vulnerability Factor (CQVF)".

Thinning with CQVF can be applied to a single name case where $S = \{s_i\}$ for some $s_i \in S^*$.

Now, we introduce a couple examples of CQVF model[4].

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[3]In order to satisfy condition (1), the following normalization is often used:

$$Z_t^i = \frac{\theta_t^i \tilde{Z}_t^i}{\sum_{j=1}^M \theta_t^j \tilde{Z}_t^j}. \tag{7}$$

Though such a normalization is not supposed for our discussion below, we mention the comparison of the estimates θ_t^S between with and without the normalization in Appendix B.

[4]As another example, Yamanaka [14] provided a piece-wise constant CQVF. $\theta_t^S = \overline{\theta}^S + \epsilon_t^S$, where $\overline{\theta}^S$ is some positive constant and ϵ_t^S is a process with piece-wise constant path. Specifically, $\epsilon_t^S = \beta_j^S$ if $H_j \leq t < H_{j+1}$ for a sequence $\{H_j\}_{j=0,1,2,\cdots}$ of increasing times, where $\{\beta_j^S\}_{j=0,1,2,\cdots}$ are independent and identically distributed random variables following the same beta distribution with mean 0.

Example 1: Constant CQVF Model The simplest is a "constant CQVF" model, in short,

$$\theta_t^S \equiv \overline{\theta}^S, \tag{8}$$

for some positive constant $\overline{\theta}^S$.

Remark In the case of the number of constituents in sub-portfolios changes, the "primitive CQVF" model, namely $\theta_t^S = 1$ ($^\forall S \subset S^*$) is derived.

Consider two sub-portfolios S_1 and S_2 where $S_1 \cup S_2 = S^*$, $S_1 \cap S_2 = \emptyset$. Also, we suppose the number of constituents in sub-portfolio S_1 increases in the period from t_1 to t_2, that is, there are t_1 and $t_2 (> t_1)$ such that $X_{t_2}^1 - X_{t_1}^1 > 0$.

Then, the following conditions should be satisfied for both t_1 and t_2 (see (1)):

$$\begin{cases} \frac{X_{t_1}^1}{X_{t_1}^*}\overline{\theta}^1 + \frac{X_{t_1}^2}{X_{t_1}^*}\overline{\theta}^2 = 1, \\ \frac{X_{t_2}^1}{X_{t_2}^*}\overline{\theta}^1 + \frac{X_{t_2}^2}{X_{t_2}^*}\overline{\theta}^2 = 1. \end{cases} \tag{9}$$

As the solution of equation (9), we obtain $\overline{\theta}^1 = \overline{\theta}^2 = 1$. This simple example indicates that non-primitive constant CQVF model assumes that the number of constituents never changes.

Example 2: Covariate CQVF Model We are able to introduce some covariates (whose number is denoted by K) for description of CQVF θ_t^S. For example, we can model θ_t^S by

$$\theta_t^S = \overline{\theta}^S \exp\left(a_1^S Y_{1,t}^S + \cdots + a_K^S Y_{K,t}^S\right), \tag{10}$$

where $\overline{\theta}^S$ is a positive constant, and $\{Y_{k,t}^S\}_{k=1,\cdots,K}$ are the covariates observed at time t and common to sub-portfolio S, and $\{a_k^S\}_{k=1,\cdots,K}$ are coefficients to be estimated. We call such a model "covariate CQVF" model.

We remark that the constant CQVF can be regarded as a special case of the covariate CQVF (10) without covariates.

Now let us consider a single name portfolio given by $S = \{s_i\}$ ($s_i \in S^*$). Then we can construct the single name intensity $\lambda_t^{s_i}$ according to CQVF model (10). For simplicity, we assume that X_t^* does not vary in time, namely $X_t^* \equiv \overline{X}^*$ for some positive integer $\overline{X}^*$.

First, we assume the total intensity λ_t^* is modeled as

$$\lambda_t^* = \exp\left(b_0^* + b_1^* Y_{1,t}^* + \cdots + b_{\tilde{M}}^* Y_{\tilde{K},t}^*\right) \tag{11}$$

where $\{Y^*_{k,t}\}_{k=1,\cdots,\tilde{K}}$ are covariates common to all the obligors in S^*, and $\{b^*_k\}_{k=0,\cdots,\tilde{K}}$ are coefficients. Such covariates $\{Y^*_{k,t}\}_{k=1,\cdots,\tilde{K}}$ would be economy-wide factors, for example, treasury rate, GDP, unemployment rate, and so on.

Next, it follows from (3) and $Z^{s_i}_t$ specified by CQVF model θ^S_t in (10) that the single name intensity $\lambda^{s_i}_t$ for sub-portfolio $\{s_i\}$ can be described as

$$\begin{aligned}\lambda^{s_i}_t &= Z^{s_i}_t \lambda^*_t = \frac{X^{s_i}_t}{X^*_t}\theta^{s_i}_t \lambda^*_t \\ &= \frac{1}{\overline{X}^*}\overline{\theta}^{s_i} \exp\left(a^{s_i}_1 Y^{s_i}_{1,t} + \cdots + a^{s_i}_K Y^{s_i}_{K,t}\right)\lambda^*_t \\ &= \tilde{\theta}^{s_i}_0 \exp\left(a^{s_i}_1 Y^{s_i}_{1,t} + \cdots + a^{s_i}_K Y^{s_i}_{K,t} + b^*_1 Y^*_{1,t} + \cdots + b^*_{\tilde{K}} Y^*_{\tilde{K},t}\right) \end{aligned} \qquad (12)$$

where $\tilde{\theta}^{s_i}_0 = \dfrac{\overline{\theta}^{s_i} e^{b^*_0}}{\overline{X}^*}$ and $\{Y^{s_i}_{k,t}\}_{k=1,\cdots,K}$ are covariates of personal information of obligor s_i, whose examples can be some financial indicators calculated from accounting information or "distance to default" dependent on market information such as individual stock prices.

This is consistent with the individual intensity model suggested by Duffie et al.[1].

As an illustration, we show some estimation results of constant CQVF of industrial category portfolio and its application to portfolio credit risk analysis in the next section. The primitive thinning model of $\theta^S_t = 1$ ($^\forall S \subset S^*$) is the thinning model employed in Giesecke and Kim [5] and Yamanaka et al. [15, 16].

4 Application: analyses of credit rating transition of Japanese firms with constant CQVF

In this section, we use some historical data on credit rating transition of Japanese firms to estimate constant CQVF (see Example 1 of the previous section) for several sub-portfolios[5]. Then, we detect the frequency of credit

[5] As we remarked in the previous section, the only allowable constant model is the primitive model with $\theta^S_t = 1$ if the number of constituents in sub-portfolios can change in time. However we try to estimate CQVF with the constant model even for such portfolios consisting of variable constituents. The one reason why we employ the constant CQVF model is that it is enough to learn whether the CQVF can be useful for improving the accuracy of thinning. As other reasons, we can regard the constant CQVF as an approximation of the time-varying CQVF if the number of constituents is less fluctuated, and

event occurrence of each sector. Also, we calculate the number of downgrades occurs in several portfolios and see the effect of the portfolio constituent sectors ratio.

4.1 Estimation Procedure

We mention the estimation for the thinning model with covariate CQVF (10).

Suppose that the economy-wide portfolio S^* consists of M disjoint non-empty sub-portfolios $S_1, \cdots, S_M$.

The data for model estimation consists of pairs of when an event happens and in which portfolio the event occurs.

Specifically, the data available at time t, denoted by $\mathcal{H}_t$ can be represented by the family of event times observed and in which sub-portfolio each observed event occurs:

$$\mathcal{H}_t = \{(T_n, \mathcal{T}^1 \cap [0,t], \cdots \mathcal{T}^M \cap [0,t])\}_{T_n \le t}.$$

Let us remember that $\mathcal{T}^S$ is the set of stopping times at which credit events occur in sub-portfolio S.

In general, because of the equality constraint (1), the CQVFs θ^m can be freely given for only $(M-1)$ sub-portfolios and that of the last one is determined by the equality constraint (1).

We suppose that $\theta^1_t, \cdots, \theta^{M-1}_t$ are modeled as the covariate CQVF model[6] and the thinning for sub-portfolio S_M is calculated via the equality constraint (1). Also we assume that the parameters to be estimated are $\Theta = \{(\overline{\theta}^m, a^m_1, \cdots, a^m_K)\}_{m=1,\cdots,M-1}$.

In ordet to estimate parameters Θ of the covariate CQVF model (10) for sub-portfolio $S_1, \cdots, S_{M-1}$, we employ a maximum likelihood method.

Since $\{Z^m_{T_n}\}_{n \le N_t, m=1,\cdots,M}$ are independent given the data $\mathcal{H}_t$ and $Z^S_{T_n} = P(T_n \in \mathcal{T}^S \mid T_n \in \mathcal{T}^*, \Theta)$ for any sub-portfolio S, the likelihood function is

$$L(\Theta \mid \mathcal{H}_t) = \prod_{m=1}^{M-1} \prod_{T_n \le t | T_n \in \mathcal{T}^m} Z^m_{T_n} \times \prod_{T_n \le t | T_n \in \mathcal{T}^M} \left(1 - \sum_{m=1}^{M-1} Z^m_{T_n}\right). \qquad (13)$$

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we can expect to see the average of time-varying CQVF at least even if the number of constituents is sharply fluctuated.

[6]We remark that the estimation can depend on which sub-portfolios are chosen for constructing free $(M-1)$ CQVF models.

Therefore the log-likelihood function of the thinning model can be given by

$$\log L(\Theta \mid \mathcal{H}_t) = \sum_{m=1}^{M-1} \sum_{T_n \le t | T_n \in \mathcal{T}^m} \log\left(Z^m_{T_n}\right) + \sum_{T_n \le t | T_n \in \mathcal{T}^M} \log\left(1 - \sum_{m=1}^{M-1} Z^m_{T_n}\right)$$
$$= \sum_{m=1}^{M-1} \sum_{T_n \le t | T_n \in \mathcal{T}^m} \left\{\log\left(\tilde{Z}^m_{T_n}\right) + \log\left(\overline{\theta}^m\right) + \left(a^m_1 Y^m_{1,t} + \cdots + a^m_K Y^m_{K,t}\right)\right\}$$
$$+ \sum_{T_n \le t | T_n \in \mathcal{T}^M} \log\left(1 - \sum_{m=1}^{M-1} \tilde{Z}^m_{T_n} \overline{\theta}^m e^{a^m_1 Y^m_{1,t} + \cdots + a^m_K Y^m_{K,t}}\right), \qquad (14)$$

where $\tilde{Z}^S_t$ for sub-portfolio S stands for the ratio of the constituents in sub-portfolio S defined in (5).

As for the empirical study below, we use the much simpler constant CQVF model for two partitions of the economy-wide portfolio with the log-likelihood function (15)[7].

Then we need to estimate only one parameter $\overline{\theta}^S$ of the one target sub-portfolio S, so the log-likelihood function is given by

$$\log L(\overline{\theta}^S \mid \mathcal{H}_t) = \sum_{T_n \le t | T_n \in \mathcal{T}^S} \left\{\log \tilde{Z}^S_{T_n} + \log \overline{\theta}^S\right\} + \sum_{T_n \le t | T_n \notin \mathcal{T}^S} \log\left(1 - \tilde{Z}^S_{T_n} \overline{\theta}^S\right). \qquad (15)$$

4.2 Estimation of constant CQVFs

In this subsection, we use some historical data on credit rating transition of Japanese firms to estimate the constant CQVF of the sub-portfolios that partition the economy-wide portfolio. We presume that the economy-wide portfolio is partitioned into five sub-portfolios according to the industrial sector defined as Tokyo Stock Exchange (TSE) 33 sector classification We give the relations between the five sub-portfolios and TSE 33 sector classification as seen in Table 1.

As the sample data of credit events, we use the historical records on downgrade and upgrade from January 4, 2000 to December 30, 2010 of the Japanese firms, which were announced by Rating and Investment Information, Inc. (R&I)[8]. Thus we implicitly suppose that the economy-wide (EW)

[7] As we can see in Appendix A, the estimates obtained every sector with the simpler two partition log-likelihood function (15) are not so different from those with the entire log-likelihood function (14).

[8] Rating and Investment Information is one of the largest credit rating agency in Japan.

Table 1: The relations between the five sub-portfolios and Tokyo Stock Exchange 33 sector classification.

5 sub-portfolios	33 Sector Classification	
Cyclical	Mining	Textiles & Apparels
	Pulp & Paper	Chemicals
	Oil & Coal Products	Rubber Products
	Glass & Ceramics Products	Iron & Steel
	Nonferrous Metals	Metal Products
	Marine Transportation	Wholesale Trade
Defensive	Fishery, Agriculture & Forestry	Foods
	Pharmaceutical	Electric Power
	Gas	
High Technology	Machinery	Electric Appliances
	Transportation Equipment	Precision Instruments
	Information & Communication	
Financial	Securities & Commodity Futures	Banks
	Life Insurance	Non-life insurance
	Other Financing Business	
Domestic	Construction	Other Products
	Land Transportation	Air Transportation
	Warehousing & Harbor	Retail Trade
	Real Estate	Services
	Transportation Services	

portfolio S^* is constituted by all the firms where R&I provided the credit rating each period. As such, each of the five partitioning sub-portfolios ("Cyclical", "Defensive", "High Tech.", "Financial", and "Domestic") consists of the firms that have (had) some credit rating provided by R&I.

We use such records on downgrade and upgrade to estimate CQVFs of downgrade and upgrade separately for each sub-sector with the simple ML with the log-likelihood function (15). The estimation results are displayed in Table 2.

As discussed in section 3, if the estimate satisfies the inequality $\overline{\theta}^S > 1$ (resp. $\overline{\theta}^S < 1$), the sub-portfolio S can be considered as relatively vulnerable (resp. stable) in credit quality. "High" (resp. "Low") in the column of "Event Frequency" in Table 2 indicates that satisfied is not only $\overline{\theta}^S > 1$ (resp. $\overline{\theta}^S < 1$) but also the absolute value of the difference between the estimate and the null hypothsis $\overline{\theta}^S = 1$ is larger than twice of the standard error given in the parenthesis below the estimate (meaning about 5% significance level).

Table 2 indicates that "Defensive" is the lowest rating-change frequency sector while "Financial" is the highest. In short, indeed the credit qualities of "Financial" is likely to be more vulnerable than "Defensive" sub-portfolio. In addition, Table 2 implies that downgrades in 'Cyclical" and upgrades in "Domestic" can be less frequent for the component proportion ratio to the whole portfolio.

Next we calculate AIC for CQVF model as well as the primitive model ($\overline{\theta}^S = 1$) in Table 3.

From the principle that the model with smaller AIC should be selected, Table 3 implies that the constant CQVF model rather than the primitive model seems more appropriate for "Cyclical", "Defensive" and "Financial" in case of downgrades, while so does for "Defensive", 'Financial" and "Domestic" in case of upgrades.

Moreover we examine the closeness between the monthly expected number implied by the model and the actual monthly observations of downgrades/upgrades so as to compare the model prediction power between the constant CQVF model and the primitive one.

For the purpose, we compute the conditional expectations of credit event counts happened per month for each sector S_i given the data available up to the end of the current month as well as the event times in the next one month. Specifically, if t_m and t_{m+1} are respectively the end of the current month and that of the next month, the conditional expectation of $N^i_{t_{m+1}} - N^i_{t_m}$ can be

Table 2: The estimates of the constant CQVF for the five sub-portfolios the simple ML with the log-likelihood function (15). The values in parentheses are the standard errors. “High” (resp. “Low”) stands for $\theta > 1$ (resp. $\theta < 1$) with about 5% significance level.

	Down-Grade		Up-Grade	
Sub-port.	CQVF	Event Frequency	CQVF	Event Frequency
Cyclical	0.780	Low	1.075	-
	(0.069)		(0.087)	
Defensive	0.617	Low	0.638	Low
	(0.097)		(0.109)	
High-tech	1.092	-	1.163	-
	(0.080)		(0.090)	
Financial	1.354	High	1.336	High
	(0.115)		(0.117)	
Domestic	1.083	-	0.733	Low
	(0.069)		(0.068)	

Table 3: AIC of the constant CQVF model and the primitive model ($\overline{\theta}^S = 1$) for each sub-portfolio.

	Down-grade		Up-grade	
Sector	Primitive	CQVF	Primitive	CQVF
Cyclical	535.92	529.01	520.73	521.96
Defensive	289.05	279.50	242.04	235.69
High-tech	629.94	630.58	540.55	539.06
Financial	560.20	551.17	513.70	506.13
Domestic	693.43	693.94	478.49	467.17

achieved due to the argument in Section 2 by

$$\begin{aligned}
&\mathbf{E}\left[N^i_{t_{m+1}} - N^i_{t_m} \;\middle|\; \mathcal{H}_{t_m}, \{T_n\} \subset (t_m, t_{m+1}] \cap \mathcal{T}^*\right] \\
&\quad = \sum_{T_n \in (t_m, t_{m+1}] \cap \mathcal{T}^*} \mathbf{E}\left[Z^i_{T_n} \;\middle|\; \mathcal{H}_{t_m}\right] = \sum_{T_n \in (t_m, t_{m+1}] \cap \mathcal{T}^*} \tilde{Z}^i_{T_n} \cdot \widehat{\overline{\theta}^i}, \qquad (16)
\end{aligned}$$

where $\tilde{Z}^i_t$ is the ratio of the number of constituents in sub-portfolio S_i to S^* defined in (5), and $\widehat{\overline{\theta}^i}$ is the estimate of the constant CQVF $\overline{\theta}^i$ via the simple ML with (15).

Table 4 (resp. Table 5) shows some basic statistics of the difference between the expected number (16) of monthly downgrades (resp. upgrades) counts and the observation during January 2000 to December 2010 for each of the five sub-portfolios.

In addition, Figure 1 displays the time series transition of the observations of monthly downgrades as well as the conditional expectations (16) of monthly downgrade counts implied respectively by the CQVF model and the primitive model for "Cyclical" sub-portfolio.

Both tables and the figure indicate that the estimation by CQVF model is closer on average to the observations than primitive model during most of the periods.

However, we can also realize that the primitive model leads to better estimation of monthly rating changes than CQVF model when the events happen much more than usual, for example, around 2002 as seen in Figure 1. The consequence implies that some time-varying CQVF models would be more adequate for dynamic random thinning during a long period than the constant CQVF model.

4.3 Simulation analysis on portfolio credit risk in consideration of CQVF

In this subsection, we examine with some numerical simulation how much the industrial sector constitution ratio in a portfolio influences the portfolio downgrade risk.

Suppose that the economy-wide (EW) portfolio S^* is constituted by 567 bond issuers, which R&I provided the ratings on January 4, 2011, and that S^* has the distribution over the five sub-portfolios as seen in Table 6. Then we consider three hypothetical portfolios named portfolio A, B and C for downgrade simulation analysis as below.

As presented in Table 6, the three hypothetical portfolios uniformly consist of 200 bond issuers and are respectively specified as follows:

Table 4: Some basic statistics on the difference between the expected number (16) of monthly *downgrades* obtained from both CQVF and Primitive and the monthly observation during January 2000 to December 2010 for each of the five sub-portfolios.

	Cyclical		Defensive		High-tech	
	Primitive	CQVF	Primitive	CQVF	Primitive	CQVF
average	0.23	0.01	0.18	0.00	-0.09	0.00
min	-4.33	-4.92	-5.76	-6.62	-3.16	-3.00
5%-percentile	-1.31	-1.61	-0.89	-0.94	-1.86	-1.76
10%-percentile	-0.78	-1.04	-0.54	-0.71	-1.34	-1.28
25%-percentile	-0.08	-0.40	0.00	0.00	-0.42	-0.27
median	0.14	0.00	0.22	0.13	0.00	0.00
75%-percentile	0.69	0.53	0.47	0.29	0.37	0.49
90%-percentile	1.26	0.82	1.01	0.63	0.91	1.00
95%-percentile	2.00	1.45	1.17	0.72	1.28	1.47
max	4.02	2.91	2.66	1.64	2.97	3.43

	Financial		Domestic	
	Primitive	CQVF	Primitive	CQVF
average	-0.22	0.00	-0.10	0.00
min	-8.06	-7.02	-4.46	-4.25
5%-percentile	-2.65	-2.16	-1.88	-1.79
10%-percentile	-1.42	-1.14	-1.40	-1.25
25%-percentile	-0.35	-0.01	-0.65	-0.49
median	0.14	0.22	0.00	0.00
75%-percentile	0.42	0.58	0.52	0.56
90%-percentile	0.70	0.98	1.09	1.18
95%-percentile	0.90	1.31	1.64	1.81
max	1.65	2.24	3.25	3.77

Table 5: Some basic statistics on the difference between the expected number (16) of monthly *upgrades* obtained from both CQVF and Primitive and the monthly observation during January 2000 to December 2010 for each of the five sub-portfolios.

	Cyclical		Defensive		High-tech	
	Primitive	CQVF	Primitive	CQVF	Primitive	CQVF
average	-0.06	0.00	0.14	0.00	-0.13	0.00
min	-4.86	-4.70	-1.29	-1.55	-2.97	-2.65
5%-percentile	-1.66	-1.56	-0.80	-0.90	-1.88	-1.62
10%-percentile	-1.04	-0.96	-0.65	-0.80	-1.34	-1.09
25%-percentile	-0.40	-0.34	0.00	0.00	-0.56	-0.48
median	0.02	0.10	0.12	0.07	0.00	0.00
75%-percentile	0.44	0.50	0.42	0.26	0.45	0.53
90%-percentile	0.71	0.78	0.75	0.48	0.89	1.06
95%-percentile	1.26	1.39	1.06	0.67	1.25	1.50
max	2.29	2.46	1.39	0.88	2.28	2.65

	Financial		Domestic	
	Primitive	CQVF	Primitive	CQVF
average	-0.20	-0.01	0.26	0.00
min	-7.96	-7.28	-2.18	-2.40
5%-percentile	-2.60	-2.33	-1.26	-1.58
10%-percentile	-1.58	-1.13	-0.89	-1.25
25%-percentile	-0.54	-0.16	-0.17	-0.56
median	0.10	0.17	0.28	0.00
75%-percentile	0.30	0.48	0.61	0.41
90%-percentile	0.70	1.08	1.41	0.97
95%-percentile	1.04	1.45	1.86	1.33
max	2.71	3.63	3.40	2.49

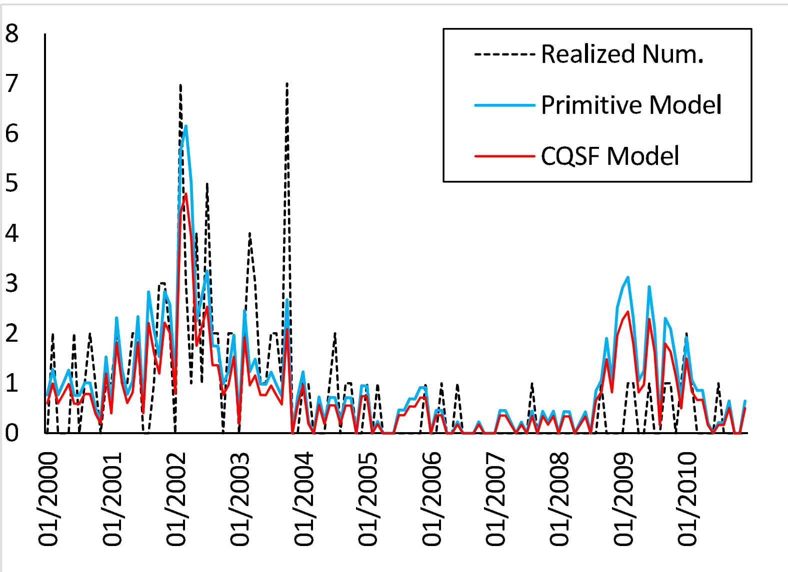

Figure 1: The time series transition of the observations of monthly downgrades as well as the conditional expectations (16) of monthly downgrade counts implied respectively by the CQVF model and the primitive model for "Cyclical" sub-portfolio.

- Port. A is over-weighted to “Cyclical” and “Defensive”, whose downgrade CQVFs are relatively low.
- Port. B is almost equally distributed over the five sub-portfolios introduced in the previous subsection.
- Port. C is over-weighted to “Financial” and “Domestic”, whose downgrade CQVFs are relatively high.

Remark that we do not need to identify the firms' names in each portfolio for the simulation since we use only the CQVF of each sub-portfolio without each firm' property.

Table 6: The distributions over the five sub-portfolios of the economy-wide (EW) portfolio S^* as well as three hypothetical portfolios for downgrade simulation analysis. There are 567 bond issuers which R&I provided the rating on January 4, 2011.

Sector	EW	Port.A	Port.B	Port.C
Cyclical	122	122	43	0
Defensive	60	60	21	0
High-tech	124	0	44	92
Financial	108	0	38	108
Domestic	153	18	54	0
Total	567	200	200	200

Assume that we have to estimate a downgrade risk for each hypothetical portfolio from January 4, 2011 to December 30, 2011. Then the model parameters are estimated with the historical data from January 4, 2000 to December 30, 2010. We begin to specify and estimate the economy-wide intensity model and then generate many scenarios of downgrades by random thinning associated with the constant downgrade CQVFs.

We firstly specify the economy-wide intensity model as

$$\lambda_t^* = \exp(\alpha + \beta Y_t), \tag{17}$$

to estimate the parameters α and β, where Y_t is some single covariate.

For a tentative illustration, we suppose that the covariate Y_t stands for the annual return at time t of TOPIX (TOkyo stock Price IndeX) [9].

[9]We calculate the realized annual return $\tilde{y}_t$ at time t as 250 business day return, that

We employ the maximum-likelihood estimation with the samples observed from January 4, 2000 to December 30, 2010 to obtain $\hat{\alpha} = 3.81$ (0.04) and $\hat{\beta} = -2.13$ (0.17), where the standard errors are the values in parentheses.

The estimation procedures of the intensity is explained in Appendix C.

We remark that the negative value of $\hat{\beta}$ implies that downgrades are more (resp. less) likely when TOPIX has fallen (resp. risen) from one year before. It is consistent with naive intuition if seeing TOPIX as a proxy of the whole economic condition.

The posterior downgrade intensity paths of the economy-wide portfolio and the five sub-portfolios are presented in Figure 2. The path of the posterior economy-wide portfolio downgrade intensity $\widehat{\lambda_t^*}$ is drawn due to (17), namely, $\widehat{\lambda_t^*} = \exp(\hat{\alpha} + \hat{\beta}\tilde{y}_t)$, where $\tilde{y}_t$ is the realized annual TOPIX return at time t. The downgrade intensity paths of the five sub-portfolios are obtained by thinning ((3) and (6)) of the posterior economy-wide portfolio downgrade intensity $\widehat{\lambda_t^*}$ with the CQVF thinning model whose constant CQVF estimate $\overline{\theta}^i$ is are shown in Table 15.

We generate 100,000 scenarios of downgrade events by Monte Carlo simulation on random thinning associated with the constant downgrade CQVF for the five sub-portfolios estimated in the previous subsection. In order to obtain samples of credit event times, we firstly obtained the intensity of each sector by the thinning model (3) and (6). Then, we simulate event times with each sector-based intensity: the event times of sector S^i on k-th business day, $(t_k = T_0^k <)T_1^k < T_2^k < \ldots (< t_{k+1})$, are obtained as $T_{j+1}^k - T_j^k = -\dfrac{\log U_k}{\lambda_{t_k}^i}$, where $\{U_k\}$ are a series of (0, 1)-uniform random variables. This procedures are based on the simulation procedure of stationary Poisson process.

For the intensity simulation on the out-of-sample period, we use TOPIX annual return on the out-of-sample period, from January 4, 2011 to December 30, 2011, to obtain the whole intensity $\widehat{\lambda_t^*}$ on the period. Table 7 stands for the expected downgrade counts for one year each portfolio that is obtained by the simulation. Also, Figure 3 displays the distributions of simulated downgrade counts for one year each portfolio.

Both Table 7 and Figure 3 show that downgrades can occur the most frequently in Port.C among the three portfolios, while there are fewer down-

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is, we obtain annual return of TOPIX by

$$\frac{S_t - S_{t-250\text{business days}}}{S_{t-250\text{business days}}}, \tag{18}$$

where S_t and $S_{t-250\text{business days}}$ are the closing price of TOPIX at the date t and 250 business days before date t respectively.

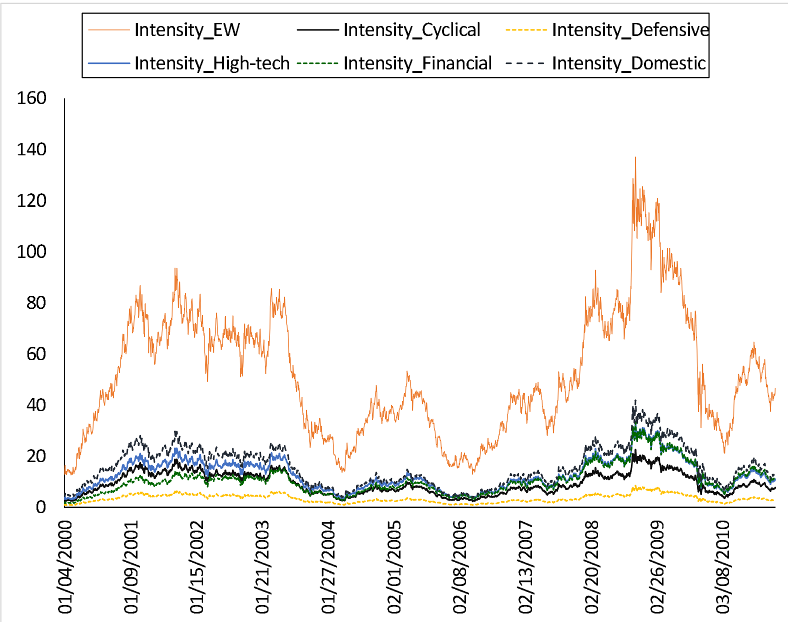

Figure 2: The posterior downgrade intensity paths of the economy-wide portfolio and the five sub-portfolios. The economy-wide intensity path is obtained by $\widehat{\lambda}_t^* = \exp(3.81 - 2.13\tilde{y}_t)$, where $\tilde{y}_t$ is the actual annual return at time t of TOPIX. The intensity of each sector is obtained by the thinning model (3) and (6) with the constant CQVF.

grades in Port.A. The consequence is quite natural as expected in advance.

On the other hand, as the primitive model is independent of portfolio sector distribution, the simulated distributions of downgrade frequency are almost the same over all the portfolios.

Table 7: Some statistics of the downgrade frequency distributions obtained by Monte Carlo simulations with 100,000 trials. Figures correspond to the number of downgrade events generated by the simulations.

		Port A	Port B	Port C
CQVF Model	Average	13.9	18.7	22.6
	99-percentile	23	29	34
	Max	32	40	44
Primitive Model	Average	18.3	18.3	18.3
	99%-percentile	29	29	29
	Max	40	38	40

5 Concluding remarks

In this paper, we suggest introducing the notion of credit quality vulnerability factor (CQVF) to thinning of the economy-wide credit event intensity in the top-down approach in order to improve credit risk measurement of sub-portfolios.

Although we just use a simple constant CQVF model to analyze some sub-portfolios classified according to types of industry with the historical data on credit rating changes of the Japanese firms, the estimation results directly indicate which sub-factors are more (less) vulnerable for credit rating changes by and large. Therefore we can conclude that thinning with some CQVF is more useful to estimate the future frequency of credit rating changes in sub-portfolios than primitive thinning with a naive composition rate of the sub-portfolios in the whole. Moreover, through a simulation analysis of some hypothetical portfolios, we can see the relation between the distribution of CQVF associated with the portfolios and the portfolio credit risk.

For our further work, we will develop the top-down credit risk framework with more sophisticated random thinning in accuracy and tractability for practical use of risk management.

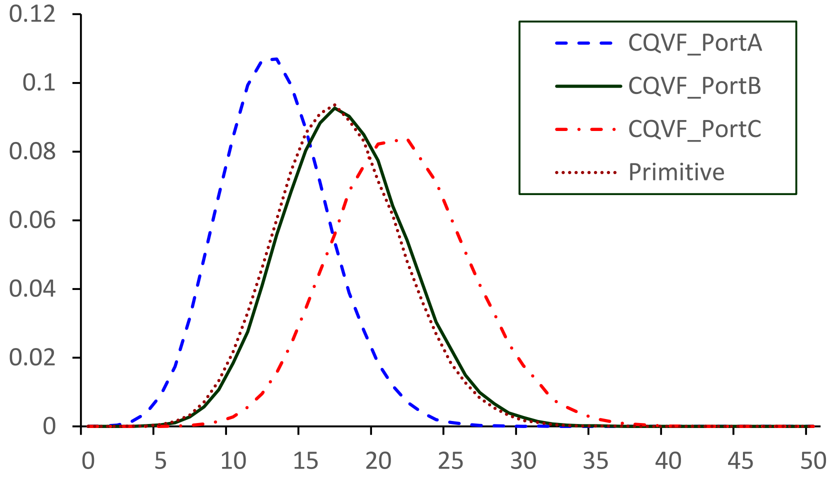

Figure 3: Distributions of simulated downgrade counts in one year each portfolio.

A Comparative analysis on estimation procedures

In section 4 we employ the maximum likelihood (ML) method with (15), which respectively estimates the CQVF of each sub-portfolios, although it appears natural to simultaneously estimate the CQVFs of all the sub-portfolios by ML with the log-likelihood (14).

Here we examine the difference of the estimates between the two methods. Remember the settings with the five sector sub-portfolios introduced in section 4.

For the regular ML with the log-likelihood (14), we regard the domestic sector S_5 as the sub-portfolio whose CQVF $\widehat{\overline{\theta}}^5$ cannot be directly estimated. In order to obtain $\widehat{\overline{\theta}}^5$, we firstly have the time series given by

$$\widehat{\overline{\theta}}^5_{T_n} = \frac{1 - \sum_{i=1}^4 \tilde{Z}^i_{T_n} \widehat{\overline{\theta}}^i}{\tilde{Z}^5_{T_n}}, \ T_n \in \mathcal{T}^*, \tag{19}$$

where $\{\overline{\theta}^i\}_{i=1,\cdots,4}$ are the constant CQVF estimates for sub-portfolios S_1, S_2, S_3

and S_4 obtained directly by ML with (14). Then we obtain the CQVF $\widehat{\overline{\theta}^5}$ of the domestic sector S_5 as the average in time of $\{\widehat{\overline{\theta}^5_{T_n}}\}_{n=1,\cdots,N_t}$, namely,

$$\widehat{\overline{\theta}^5} = \sum_{n=1}^{N_t} \frac{\widehat{\overline{\theta}^5_{T_n}}}{N_t}. \tag{20}$$

The estimation result is displayed in table 8. Table 8 implies that the estimates by the simpler ML with (15) are not so far in terms of standard errors from the regular ML with (14).

Hence we consider that we can use the simple two partition ML with (15) for our empirical study.

Table 8: The ML Estimates of CQVF for the five sectors by the simpler two-partition ML (Simple) with (15) as well as the regular ML estimation (Regular) with (14) and (19)-(20). The values in parentheses are the standard errors.

	Down-Grade		Up-Grade	
Sector	Simple	Regular	Simple	Regular
Cyclical (Sec1)	0.780	0.792	1.075	1.085
	(0.069)	(0.062)	(0.087)	(0.068)
Defensive (Sec2)	0.617	0.624	0.638	0.647
	(0.097)	(0.092)	(0.109)	(0.099)
High-tech (Sec3)	1.092	1.092	1.163	1.163
	(0.080)	(0.069)	(0.090)	(0.069)
Financial (Sec4)	1.354	1.294	1.336	1.323
	(0.115)	(0.095)	(0.117)	(0.090)
Domestic (Sec5)	1.083	1.095	0.733	0.740
	(0.069)	-	(0.068)	-

B Normalization of thinning processes

If the partitioned sub-portfolios $S_1, \cdots, S_M$ of S^* are simultaneously considered, the CQVF models should satisfy the condition (1) that the sum of thinning processes is always one. Thus, we often consider the normalized thinning process specified by

$$Z_t^i = \frac{\theta_t^i \tilde{Z}_t^i}{\sum_{j=1}^M \theta_t^j \tilde{Z}_t^j} = \tilde{\theta}_t^i \tilde{Z}_t^i, \tag{21}$$

where we call $\tilde{\theta}_t^i = \dfrac{\theta_t^i}{\sum_{j=1}^M \theta_t^j \tilde{Z}_t^j}$ "the normalized CQVF." However section 4 does not particularly assume such a normalization of CQVF for simplicity.

Here we compare the estimation of CQVF with normalization with that of unnormalized case appeared in section 4.

For the estimation of CQVF with normalization, we assume the pre-CQVF θ_t^i is constant in (21), namely $\theta_t^i = \overline{\theta}^i$. Then we obtain the estimates $\widehat{\overline{\theta}}^i$ of the pre-CQVF via ML estimation with log-likelihood function given by

$$\begin{aligned}\log L(\Theta \mid \mathcal{H}_t) &= \sum_{m=1}^M \sum_{T_n \le t | T_n \in \mathcal{T}^m} \log \left(Z_{T_n}^m \right) \\ &= \sum_{m=1}^M \sum_{T_n \le t | T_n \in \mathcal{T}^m} \left\{ \log \left(\tilde{Z}_{T_n}^m \right) + \log \left(\overline{\theta}^m \right) \right\}.\end{aligned}$$

The estimates $\widehat{\overline{\theta}}^i$ of the pre-CQVF from the samples presented in section 4 are shown in Table 9.

Table 9: ML estimates of the pre-normalized CQVF $\widehat{\overline{\theta}}^i$. The values in parentheses are standard errors.

Sector	Down-Grade	Up-Grade
Cyclical (Sec 1)	1.239	1.418
	(0.135)	(0.153)
Defensive (Sec 2)	0.988	0.842
	(0.166)	(0.154)
High-tech (Sec 3)	1.749	1.534
	(0.171)	(0.161)
Financial (Sec 4)	2.174	1.794
	(0.232)	(0.202)
Domestic (Sec 5)	1.734	0.966
	(0.159)	(0.112)

After that, with the pre-normalized CQVF estimators, we calculate the normalized CQVF estimators at each event time as follows:

$$\widehat{\tilde{\theta}^i_{T_n}} = \frac{\widehat{\overline{\theta}^i_{T_n}}}{\sum_{j=1}^{M} \widehat{\overline{\theta}^j_{T_n}} \tilde{Z}^j_{T_n}}, \quad T_n \in \mathcal{T}^*.$$

In order to compare the normalized CQVF with the unnormalized shown in section 4, we calculate the time-series average $\widehat{\tilde{\theta}}^i$ of the normalized CQVF:

$$\widehat{\tilde{\theta}}^i = \sum_{n=1}^{N_t} \frac{\widehat{\tilde{\theta}^i_{T_n}}}{N_t}.$$

Table 10 shows the CQVF estimates of the average normalized CQVF as well as the unnormalized constant CQVF estimates presented in section 4.

Table 10 naively indicates that average normalized CQVFs are not so different from the unnormalized constant CQVFs in terms of standard errors. This consequence implies that it is not serious for our empirical analysis to ignore the normalization of CQVF.

Table 10: Estimated CQVFs for the five sectors. The values in parentheses are standard errors.

	Down-Grade		Up-Grade	
Sector	Unnormalized	Normalized	Unnormalized	Normalized
Cyclical (Sec 1)	0.780	0.774	1.075	1.075
	(0.069)	-	(0.087)	-
Defensive (Sec 2)	0.617	0.616	0.638	0.638
	(0.097)	-	(0.109)	-
High-tech (Sec 3)	1.092	1.092	1.163	1.162
	(0.080)	-	(0.090)	-
Financial (Sec 4)	1.354	1.358	1.336	1.359
	(0.115)	-	(0.117)	-
Domestic (Sec 5)	1.083	1.083	0.733	0.732
	(0.069)	-	(0.068)	-

C Parameter estimation of the intensity process

We apply the maximum likelihood method in order to obtain the parameters of EW intensity model given by (17). We assume that 12:00 a.m. on January 4, 2000 is set as $t = 0$ and that the observed raw event dates $\{\tilde{T}_n\}_{n=1,\ldots,N}$ with $0 \leq \tilde{T}_1 \leq \tilde{T}_2 \leq \ldots \leq \tilde{T}_N (< H)$ are available for estimation. Also suppose that each raw event date can be represented like $\tilde{T}_n = m\Delta t$ for some integer m and $\Delta t = 0.004$, which means 250 business days amount to a unit of time.

Now the log-likelihood function for α and β in (17) can be approximated as follows.

$$\sum_{n=1}^{N} \log \lambda^*_{\tilde{T}_n-} - \int_0^H \lambda^*_s \mathrm{d}s \approx N\alpha + \beta \sum_{n=1}^{N} \tilde{y}_{\tilde{T}_n} - e^{\alpha}\Delta t \sum_{m=0}^{M-1} e^{\beta \tilde{y}_{m\Delta t}},$$

where, $\tilde{T}_0 = 0, H = M\Delta t$ for some integer M, and $\{\tilde{y}_{m\Delta t}\}_{m=0,1,\ldots,M-1}$ are the daily samples during the period $[0, H)$ of the TOPIX annual return calculated via (18).

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