HUB-FS Working Paper Series

FS-2020-E-001

A Default Contagion Model for Pricing Defaultable Bonds from An Information Based Perspective

Hidetoshi Nakagawa

School of Business Administration
Hitotsubashi University

Hideyuki Takada

Department of Information Science, Toho University

First version: September 24, 2020

Current version: July 8, 2021

A DEFAULT CONTAGION MODEL FOR PRICING DEFAULTABLE BONDS FROM AN INFORMATION BASED PERSPECTIVE

HIDETOSHI NAKAGAWA[1] AND HIDEYUKI TAKADA[2]

Abstract. In this study, we introduce an extended model of the information based model of credit risk proposed by Brody, Hughston and Macrina (2010) to a multi-name case to investigate how default contagion risk influences the price fluctuation of defaultable discount bonds. Under the model with a couple of obligors, we derive a stochastic differential equation for one defaultable zero-recovery discount bond price process to reflect default contagion risk of a counterpart debt obligor. As a consequence, we find that the excess rate of the return in the trend term of the bond consists of not only the issuer's hazard rate but also the counterpart obligor's hazard rate adjusted with the "pseudo-default loss" rate. We also find that the bond price can jump at the default time of the counterpart by the amount dependent on the correlation between the issuer and the counterpart. Moreover, we numerically examine the impact of default contagion risk on some bond price components within the model.

KEY WORDS: Default contagion; Information-based approach; Defaultable discount bond

1. Introduction

In this paper, we study how default contagion influences the price fluctuation of defaultable discount bonds by extending the market information flow-based model proposed by Brody et al. (2010) to a multi-name case. We frequently observe that default events in the market can affect the credit quality of other active companies typically in a negative way and can cause other default events in the worst case. Such a phenomenon is often referred to as credit/default contagion. Many researchers (and practitioners) take a great deal of interest in how to model credit/default contagion, since it is likely that accurate estimation of credit/default contagion enables us to improve the measurement of counterparty risk, valuation, and hedging of credit derivatives dependent on multiple names, and so on.

Various studies exist on the modeling of credit/default contagion. We roughly classify them into two categories according to whether the contagion effect is introduced exogenously or endogenously in the model. The models in one category evolved from the interacting default intensity model developed by Jarrow and Yu (2001) and Davis and Lo (2001), where the default intensities are given exogenously to

[1]School of Business Administration, Hitotsubashi University Business School, Email: hnakagawa@hub.hit-u.ac.jp
[2]Toho University, Department of Information Science, Email: hideyuki.takada@is.sci.toho-u.ac.jp

contain potential jumps due to contagion. Thus, the jump size of the intensities are viewed as the input parameters of the model. The interacting intensity models are theoretically studied by Kusuoka (1999) in terms of the measure change, and furthermore extended to a vast variety of models such as Yu (2007), Herbertsson (2007), Frey and Backhaus (2008), Bielecki et al. (2008), Bielecki et al. (2009), Zheng and Jiang (2009), and so forth. In addition, Coculescu (2017) assumes a pre-specified contagious impact exogenously, but she discusses a more general framework so that we can consider some influences of the history of defaults on credit risk evaluation.

The other category can be regarded as modeling based on the Bayesian update of the hidden state of some factors: Schönbucher and Schubert (2001), (reorganized as Subsection 10.8.4 of Schönbucher (2003)), Section 9 of McNeil et al. (2005), and Benzoni at al. (2015). In contrast, the models in this category are conceptually inspired by empirical evidence reported by Das et al. (2011), Duffie et al. (2009), and Azizpour et al. (2009). These formulations assume that the contagious jumps of credit qualities are caused by discontinuous changes in the hidden state, and then endogenously determined as an output of the model. In this sense, an application of stochastic filtering (Frey and Schmidt (2012), Elliott and Shen (2015)) would also be categorized into this group.

We aim to consider modeling default contagion from the latter standpoint. Specifically, we use the market information flow-based model first proposed by Brody et al. (2008) (reorganized as Brody et al. (2011) and extended to credit risk modeling by the same authors (Brody et al. (2010)) as a starting point. The motivation of Brody et al. (2010) is to model the "perceived" probability of default, which can fluctuate depending on the information flow representing market sentiments of default risk. The single-name case has been fully studied by Brody et al. (2010), but multi-name cases have not yet been fully investigated. If their model is successfully extended to a multi-name setup, it is likely that the contagion effects of default events can be discussed in terms of fragile market sentiments within the information-based approach.

In addition, we remark that our model does not satisfy the so-called immersion property (($\mathcal{H}$)-hypothesis) in the above studies. Therefore, we have to carefully examine how the filtrations are specified and related to the processes in the model to achieve the price dynamics of the defaultable bonds because any classical results under the immersion property cannot be directly applied. El Karoui et al. (2010) proposed the density approach to discuss generally (rigorously) the contagion under the enlargement of filtrations, and El Karoui et al. (2015) studied successive defaults within a multi-name version of the density approach. In their approach, the conditional joint density of default times entirely determines the structure of the contagion, and hence, the contagious jumps are endogenously given. From a practical perspective, Crépey et al. (2013) and Crépey and Song (2017) constructed a specific model based on a dynamic Gaussian copula for an application to counterparty risk management.

With this background, we present an extended model of the market information flow-based model proposed by Brody et al. (2010) to a multi-name case to quantitatively recognize the default contagion effect on pricing defaultable discount bonds with zero recovery. To be more specific, we obtain some general results for the conditional joint distributions of default times in the reference universe, and then for the case of two debt obligors, we successfully derive a stochastic differential equation for a defaultable zero-recovery discount bond price process to see the default contagion risk of a counterpart debt obligor. As a consequence, we succeed to clarify how default risk dependence between two obligors in our model are understandably related to the dynamics of the defaultable bond prices in terms of the martingales which are given as compensated default indicator processes for both obligors as well as Brownian motions derived from the market information flow of both obligors. Interestingly, our market information flow-based model can be viewed as a dynamic version of the so-called Kusuoka's counterexample model (c.f. Kusuoka (1999), Bielecki and Rutkowski (2002)) since the default intensities (the compensator of the default indicator processes) deduced from our model are dependent on whether the counterpart has defaulted or not. To the best of our knowledge, this is the first work in the Bayesian updating framework that shows the detailed interaction of defaultable bonds in terms of stochastic differential equations with jumps, which enables us to comprehend the dynamics as such.

More specifically, our main results are summarized as follows. We see that if neither defaults, the equation implies that the trend term (drift term) of the defaultable bond price process includes not only of the hazard rate or the credit spread of the issuer but also of the counterpart obligor's hazard rate adjusted with the "pseudo-default loss" rate, although the underlying bond does not default due to the counterpart obligor's default. After the counterpart obligor's default, the excess rate of the return in the trend term is composed of only the issuer hazard rate, but the expression of the hazard rate is different from that before the default of counterpart. Similarly if neither one defaults, a couple of Brownian motions derived from the market information flow of both obligors randomly drive the defaultable bond price process; however, after the counterpart obligor's default, the Brownian motion from the issuer's market information flow is only the driver, where the volatility term changes from that before the default of its counterpart.

Next, we observe that the bond price can jump at the default time of the counterpart obligor. The consequence is consistent with the model assumption that the information is largely updated at the counterpart default since the counterpart's market factor is exactly revealed. We also notice whether the bond price jumps upward or downward depending on the sign of the correlation parameter between both market factors. In connection with this, we can note that such negatively correlated market factors imply negative "pseudo-default loss" rate so that the trend term of the underlying bond before the counterpart default can shrink compared to when the bond is evaluated alone.

Then we show some numerical works to observe the quantitative effects of counterpart obligors' default on the model components of the issuer. Indeed, we present some numerical illustrations on the relationship between the model components and the market factor correlation as well as the upward impact of counterpart obligors' default on the time trend term of the defaultable bond price process.

From the practical point of view, it is undeniable that our model has some computational difficulties for larger reference universe. However such difficulties are common among so-called bottom-up approach models. Some previous studies (for example Herbertsson (2007)) adopt so-called top-down approaches to bypass computational difficulties caused by combinatorial processing of the default occurrence order, but their models cannot capture the idiosyncratic contagion effects that we aim to see. As such, although there are still many challenges to put it into practical use, our results and considerations arguably provide a theoretically new and useful perspective within the Bayesian updating framework for credit risk modeling.

The remainder of this paper is organized as follows. In section 2, we introduce our information-based model of default times and derive some important propositions to price contagious defaultable discount bonds. In section 3, we describe our main theorem on the stochastic differential equation that the defaultable bond price process follows, and we provide the proof of the theorem in Section 4. We present some numerical illustrations in section 5, and finally, we conclude in Section 6.

2. Model and Preliminaries

2.1. **Information-based model of default times.** Under the assumption of no arbitrage, we model a financial market that includes several defaultable instruments on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, which is rich enough to support some Brownian motions. We assume that $\mathbb{P}$ is a risk-neutral pricing measure. The pricing measure $\mathbb{P}$ cannot be uniquely specified only by the assumption of no arbitrage due to market incompleteness. In practice, however, this assumption is sufficient for the discussion that follows since one can imply some model parameters under the pricing measure by calibrating the obtained pricing model to corporate bond or credit default swap market data. In what follows, all expectations are taken under the risk-neutral pricing measure $\mathbb{P}$.

We consider $n(\in \mathbb{N})$ debt obligors and denote by $\tau_1, \cdots, \tau_n$ random times, that is, nonnegative $\mathcal{G}$-random variables representing default times of the debt obligors, respectively. According to the definition and notation for the single obligor default model of Brody et al. (2010), we assume that for each $i = 1, 2, \cdots, n$, the default time τ_i of the obligor i is modeled as

$$\tau_i := h_i^{-1}(Z_i), \tag{1}$$

where h_i is a continuous deterministic invertible increasing function with $\lim_{s\to 0} h_i(s) = -\infty$, $\lim_{s\to\infty} h_i(s) = +\infty$, and Z_i is a standard normal random variable representing some credit-related latent market factor for the obligor i. The above specification of default time is analogue to the idea that each idiosyncratic credit risk is driven by a latent normal-distributed factor in some simplified portfolio credit risk models like ASFR Model (asymptotic single factor risk model). From another point of view, τ_i is supposed to be a totally inaccessible stopping time since we assume that Z_i is not perfectly observable. In this sense, the formulation is classified into a so-called incomplete information approach such as Duffie and Lando (2001), Nakagawa (2001), Çetin et al. (2004) and Jarrow and Protter (2004) for single-name case, and Benzoni at al. (2015) for multi-name case. We remark that the market factor Z_i is informationally equivalent to the default time τ_i via the deterministic (hence completely known) function h_i. We suppose that the credit-related market factors $Z_1, \dots, Z_n$ are correlated, so they follow an n-dimensional centered correlated normal distribution.

Remark 2.1. *Our formulation is regarded as a particular case of Brody et al. (2010) that models $\tau_i := f_i(X_1, X_2, \cdots, X_n)$ with n independent random variables $X_1, X_2, \cdots, X_n$ and some n-variate function f_i.*

Next, we introduce the concept of market information flow, whereby we can explicitly describe the amount of available information associated with the credit-related market factor. We assume that market participants can only access partial information with inseparable noise. More precisely, we define the market filtration $\{\mathcal{F}_t\}$, which stands for the information available to the market participants, as shown below.

First, for each $i = 1, \dots, n$, let $\{\xi_t^i\}$ be an i-th market information process associated with the market factor Z_i, which is specified in the following form.

$$\xi_t^i := \sigma_i t Z_i + B_t^i, \quad (1 \le i \le n) \tag{2}$$

where $\sigma_i > 0$ is termed "information flow rate" (see Brody et al. (2010)), and $\{B_t^i : 1 \le i \le n\}$ is a set of n mutually independent standard Brownian motions that are independent of all the market factors $\{Z_i\}_{i=1,\dots,n}$. Then, we specify a filtration $\{\mathcal{F}_t\}$ of the whole market information except for the occurrence of defaults by

$$\mathcal{F}_t := \sigma(\xi_s^i : 0 \le s \le t, 1 \le i \le n).$$

Then, let $\{\mathcal{H}_t^i\}$ be the filtration on the obligor i's default defined by $\mathcal{H}_t^i := \sigma(\tau_i \wedge s : 0 \le s \le t)$ for all $1 \le i \le n$, and let $\{\mathcal{H}_t\}$ be the filtration of the whole default information given by $\mathcal{H}_t := \bigvee_{i=1}^n \mathcal{H}_t^i$. Finally, we define $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for any $t \ge 0$ and view the filtration $\{\mathcal{G}_t\}$ as the total information available to the market participants.

Remark 2.2. *Clearly, the model permits the existence of an $\mathcal{F}_t$-conditional joint density $a_t(t_1, \cdots, t_n)$ of $(\tau_1, \cdots, \tau_n)$ such that*

$$\mathbb{P}\left(\tau_1 > t_1, \cdots, \tau_n > t_n \mid \mathcal{F}_t\right) = \int_{t_1}^{\infty} \cdots \int_{t_n}^{\infty} a_t(v_1, \cdots, v_n) dv_1 \cdots dv_n.$$

Then, it can be seen that the paper seeks to construct a typical (representative) example of the density approach to credit risk. For the general theory of density approach, readers can refer to El Karoui et al. (2010) for a single default, and El Karoui et al. (2015) for multiple defaults.

Proposition 2.3 (Markov property)**.** *For each $i = 1, \ldots, n$, the information process $\{\xi_t^i\}$ is a Markov process with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$. Specifically, we have*

$$\mathbb{P}(\xi_t^i \leq x \mid \xi_s^i, \xi_{s_1}^i, \xi_{s_2}^i, \cdots, \xi_{s_k}^i) = \mathbb{P}(\xi_t^i \leq x \mid \xi_s^i),$$

for any collection of times $t, s, s_1, \ldots, s_k$ with $t \geq s \geq s_1 \geq s_2 \geq \cdots \geq s_k > 0$.

Proof. See Brody et al. (2010) for the case of $n = 1$. The extension to the multi-name case is straightforward. □

Remark 2.4. *$\mathcal{F}_\infty$-measurability of Z_i should be treated carefully. It follows from* (2) *that $Z_i = \frac{1}{\sigma_i}\left(\frac{\xi_t^i}{t} - \frac{B_t^i}{t}\right)$ for any $t > 0$. Hence we have*

$$Z_i = \frac{1}{\sigma_i} \lim_{t\to\infty} \left(\frac{\xi_t^i}{t} - \frac{B_t^i}{t}\right) = \frac{1}{\sigma_i} \lim_{t\to\infty} \frac{\xi_t^i}{t} \quad a.s.$$

because of the property of $\lim_{t\to\infty} B_t^i/t = 0$ a.s., so we can see that Z_i is $\mathcal{F}_\infty$-measurable. However we remark that Z_i is not $\mathcal{F}_t$-measurable for any finite $t > 0$. In other words, it is impossible to specify Z_i from observations of $\{\xi_t^i\}$ during any finite period. This argument implies that $\mathbf{1}_{\{\tau_i \leq t\}} = \mathbb{P}(\tau_i \leq t \mid \mathcal{F}_\infty) \neq \mathbb{P}(\tau_i \leq t \mid \mathcal{F}_t)$ for any $t \geq 0$, so our model does not satisfy the so-called immersion property ($(\mathcal{H})$-hypothesis).

2.2. **Defaultable bond.** A single obligor case (n=1) was studied in detail by Brody et al. (2010), while multi-name cases ($n \geq 2$) have not yet been fully investigated. We investigate information-based credit contagion effects in terms of bond price dynamics. As we see later, the defaultable bond price processes interact with each other via their trend and volatility term due to the Bayesian update of beliefs under progressive enlargement of filtration. We begin with the generalized Dellacherie formula to deal with the conditional expectation with respect to the global filtration. For notational convenience, we denote by $[n] := \{1, 2, \cdots, n\}$ a set of all obligors in our universe.

Proposition 2.5 (Generalized Dellacherie formula). *Let Y be a $\mathcal{G}$-measurable integrable random variable, then*

$$\mathbb{E}[Y|\mathcal{G}_t] = \sum_{I\subset[n]} \left\{ \prod_{i\in I} \mathbf{1}_{\{\tau_i\le t\}} \cdot \prod_{j\in[n]\backslash I} \mathbf{1}_{\{\tau_j>t\}} \cdot \frac{\mathbb{E}\Big[Y\cdot \prod_{j\in[n]\backslash I}\mathbf{1}_{\{\tau_j>t\}}\Big|\mathcal{F}_t\vee\bigvee_{i\in I}\mathcal{H}^i_\infty\Big]}{\mathbb{E}\Big[\prod_{j\in[n]\backslash I}\mathbf{1}_{\{\tau_j>t\}}\Big|\mathcal{F}_t\vee\bigvee_{i\in I}\mathcal{H}^i_\infty\Big]} \right\}.$$

Proof. See Chapter 3 of Elouerkhaoui (2017). □

Now we look at pricing of of defaultable zero-recovery discount bonds. Similar to Brody et al. (2010), throughout the paper we assume that the credit risk-free interest rate process r_t is deterministic. Hence T-maturity credit risk-free discount bond price at time t, denoted by $P_{t,T} := \exp(-\int_t^T r_u du)$, is also deterministic. It is possible to make the credit risk-free interest rate stochastic without affecting our discussion on credit risk modeling by introducing another information process as Section 2.2.2 of Yu and Rutkowski (2007). However, our main concern is modeling the default contagion risk, so we need to pay little attention to the risk-free rate dynamics.

The price at time t of a defaultable zero-recovery discount bond issued by obligor $\alpha\in[n]$ with maturity T is given by

$$D^{(\alpha)}_{t,T} := P_{t,T}\mathbf{1}_{\{\tau_\alpha>t\}}\mathbb{E}[\mathbf{1}_{\{\tau_\alpha>T\}}|\mathcal{G}_t]. \tag{3}$$

It follows from the Markov property of $\{\xi^i_t\}$, Proposition 2.5, and the property of $\mathcal{H}^i_\infty=\sigma\{Z_i\}$ that

$$\begin{aligned} D^{(\alpha)}_{t,T} &= P_{t,T}\mathbf{1}_{\{\tau_\alpha>t\}} \sum_{I\subset[n]\backslash\{\alpha\}} \left\{ \prod_{i\in I}\mathbf{1}_{\{\tau_i\le t\}} \prod_{j\in[n]\backslash(I\cup\{\alpha\})}\mathbf{1}_{\{\tau_j>t\}} \right. \\ &\qquad \left. \times \frac{\mathbb{E}\Big[\mathbf{1}_{\{\tau_\alpha>T\}}\prod_{j\in[n]\backslash(I\cup\{\alpha\})}\mathbf{1}_{\{\tau_j>t\}}\Big|\mathcal{F}_t\vee\bigvee_{i\in I}\mathcal{H}^i_\infty\Big]}{\mathbb{E}\Big[\prod_{j\in[n]\backslash I}\mathbf{1}_{\{\tau_j>t\}}\Big|\mathcal{F}_t\vee\bigvee_{i\in I}\mathcal{H}^i_\infty\Big]} \right\} \\ &= P_{t,T}\mathbf{1}_{\{\tau_\alpha>t\}} \sum_{I\subset[n]\backslash\{\alpha\}} \left\{ \prod_{i\in I}\mathbf{1}_{\{\tau_i\le t\}} \prod_{j\in[n]\backslash(I\cup\{\alpha\})}\mathbf{1}_{\{\tau_j>t\}} \right. \\ &\qquad \left. \times \frac{\mathbb{P}\Big(\{\tau_\alpha>T\}\cap\{\tau_j>t\mid j\in[n]\backslash I\}\Big|\ \{\xi^j_t\}_{j\in[n]\backslash I},\ \{Z_i\}_{i\in I}\Big)}{\mathbb{P}\Big(\{\tau_j>t\mid j\in[n]\backslash I\}\Big|\ \{\xi^j_t\}_{j\in[n]\backslash I},\ \{Z_i\}_{i\in I}\Big)} \right\} \end{aligned} \tag{4}$$

Example 2.6. *For $n=2$, the discount bond price formulas $D^{(1)}_{t,T}$ and $D^{(2)}_{t,T}$ can be reduced to the following simple expressions:*

$$D^{(1)}_{t,T} = P_{t,T}\left\{\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}\frac{\mathbb{P}(\tau_1>T,\tau_2>t\mid\xi^1_t,\xi^2_t)}{\mathbb{P}(\tau_1>t,\tau_2>t\mid\xi^1_t,\xi^2_t)} + \mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}\frac{\mathbb{P}(\tau_1>T\mid\xi^1_t,Z_2)}{\mathbb{P}(\tau_1>t\mid\xi^1_t,Z_2)}\right\},$$

$$D^{(2)}_{t,T} = P_{t,T}\left\{\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}\frac{\mathbb{P}(\tau_1>t,\tau_2>T\mid\xi^1_t,\xi^2_t)}{\mathbb{P}(\tau_1>t,\tau_2>t\mid\xi^1_t,\xi^2_t)} + \mathbf{1}_{\{\tau_1\le t,\tau_2>t\}}\frac{\mathbb{P}(\tau_2>T\mid\xi^2_t,Z_1)}{\mathbb{P}(\tau_2>t\mid\xi^2_t,Z_1)}\right\}.$$

To obtain a specific representation of $D_{t,T}^{(\alpha)}$ given in (4), we need to calculate the conditional probabilities that appear in (4). For this purpose, we present the following two propositions. In the following, we use a simplified notations such as $(z_i)_{i\in[n]}$ for $z_1,\cdots,z_n$, and $(dz_j)_{j\in[n]}$ for $dz_1dz_2\cdots dz_n$, and so on.

Proposition 2.7. *On the set $\{\tau_1 > t,\cdots,\tau_n > t\}$, that is, if no default happens until t, we have for each $\alpha\in[n]$ and for any s $(\geq t)$,*

$$\mathbb{P}\left(\{\tau_\alpha > s\}\cap\{\tau_j > t \mid j\neq\alpha\}\Big|\ \{\xi_t^j\}_{j\in[n]}\right)$$
$$= \frac{\displaystyle\int_{\mathbb{R}^n}\mathbf{1}_{\{z_\alpha>h_\alpha(s)\}}\prod_{j\neq\alpha}\mathbf{1}_{\{z_j>h_j(t)\}}p_0\big((z_j)_{j\in[n]}\big)\exp\left(\sum_{i=1}^n\Big(\sigma_i z_i\xi_t^i-\frac{t}{2}\sigma_i^2z_i^2\Big)\right)(dz_j)_{j\in[n]}}{\displaystyle\int_{\mathbb{R}^n}p_0\big((z_j)_{j\in[n]}\big)\exp\left(\sum_{i=1}^n\Big(\sigma_i z_i\xi_t^i-\frac{t}{2}\sigma_i^2z_i^2\Big)\right)(dz_j)_{j\in[n]}},$$

where $p_0\big((z_j)_{j\in[n]}\big)= p_0(z_1,\ldots,z_n)$ is the unconditional joint density of the credit-related market factors $(Z_1,\ldots,Z_n)$, that is, the joint density of a correlated normal distribution with zero mean and unit variance.

Proof. On the set $\{\tau_1 > t,\cdots,\tau_n > t\}$, we see

$$\mathbb{P}\left(\{\tau_\alpha > s\}\cap\{\tau_j > t \mid j\neq\alpha\}\Big|\ \{\xi_t^j\}_{j\in[n]}\right) = \mathbb{P}\left(\{Z_\alpha > h_\alpha(s)\}\cap\{Z_j > h_j(t)\mid j\neq\alpha\}\Big|\ \{\xi_t^j\}_{j\in[n]}\right)$$
$$= \int_{\mathbb{R}^n}\mathbf{1}_{\{z_\alpha>h_\alpha(s)\}}\prod_{j\neq\alpha}\mathbf{1}_{\{z_j>h_j(t)\}}\pi_t\big((z_j)_{j\in[n]}\big)(dz_j)_{j\in[n]},$$

where $\pi_t\big((z_j)_{j\in[n]}\big)$ denotes the conditional joint density of $(Z_j)_{j\in[n]}$ given $\{\xi_t^j\}_{j\in[n]}$. From the Markov property of $\{\xi_t^i\}$, it can be rewritten as

$$\pi_t\big((z_j)_{j\in[n]}\big)(dz_j)_{j\in[n]} = \mathbb{P}\big(\{Z_j\in dz_j\}_{j\in[n]}\mid\{\xi_t^j\}_{j\in[n]}\big).$$

Furthermore, the Bayes formula implies that

$$\mathbb{P}\big(\{Z_j\in dz_j\}_{j\in[n]}\mid\{\xi_t^j\}_{j\in[n]}\big) = \frac{\mathbb{P}\big(\{\xi_t^j\}_{j\in[n]}\mid\{Z_j=z_j\}_{j\in[n]}\big)p_0\big((z_j)_{j\in[n]}\big)(dz_j)_{j\in[n]}}{\displaystyle\int_{\mathbb{R}^n}\mathbb{P}\big(\{\xi_t^j\}_{j\in[n]}\mid\{Z_j=z_j\}_{j\in[n]}\big)p_0\big((z_j)_{j\in[n]}\big)(dz_j)_{j\in[n]}} \tag{5}$$

where $p_0\big((z_j)_{j\in[n]}\big)$ is the prior density of $(Z_j)_{j\in[n]}$. We remark that $\xi_t^i|_{Z_i=z_i}$ and $\xi_t^j|_{Z_j=z_j}$ are (conditionally) independent if $i\neq j$, as $\{B_t^j\}_{j\in[n]}$ are mutually independent Brownian motions. Hence we have

$$\xi_t^j\big|_{Z_j=z_j}\ \sim\ N\big(\sigma_j t z_j\ ,\ t\big),\quad j=1,2,\cdots,n.$$

Then, the likelihood is obtained as

$$\mathbb{P}\big(\{\xi_t^j\}_{j\in[n]} \mid \{Z_j = z_j\}_{j\in[n]}\big) = \frac{1}{(2\pi t)^{\frac{n}{2}}}\exp\left(-\frac{1}{2t}\sum_{i=1}^{n}\big(\xi_t^i - \sigma_i t z_i\big)^2\right). \tag{6}$$

Inserting (6) into (5) yields

$$\pi_t\big((z_j)_{j\in[n]}\big) = \frac{p_0\big((z_j)_{j\in[n]}\big)\exp\left(\sum_{i=1}^{n}\Big(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2\Big)\right)}{\int_{\mathbb{R}^n} p_0\big((z_j)_{j\in[n]}\big)\exp\left(\sum_{i=1}^{n}\Big(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2\Big)\right)(dz_j)_{j\in[n]}},$$

and assertion follows. □

In addition, let $\mathcal{I}_t := \{i \in [n] : \tau_i \le t\}$ be a set of defaulted obligors up to time t, and $\mathcal{J}_t := [n]\backslash\mathcal{I}_t$ a set of surviving obligors at time t. Specifically, rearrange the order of the obligors so that the elements in $\mathcal{I}_t$ come after those in $\mathcal{J}_t$ whenever a default occurs.

Proposition 2.8. *Suppose that $\alpha \in \mathcal{J}_t$. Then, for any s $(\ge t)$,*

$$\begin{aligned}&\mathbb{P}\left(\{\tau_\alpha > s\} \cap \{\tau_j > t \mid j \in \mathcal{J}_t\backslash\{\alpha\}\}\middle|\ \{\xi_t^j\}_{j\in\mathcal{J}_t},\ \{Z_i\}_{i\in\mathcal{I}_t}\right)\\ &= \frac{\int_{\mathbb{R}^{|\mathcal{J}_t|}} \mathbf{1}_{\{z_\alpha > h_\alpha(s)\}} \prod_{j\in\mathcal{J}_t\backslash\{\alpha\}} \mathbf{1}_{\{z_j > h_j(t)\}} p\big((z_j)_{j\in\mathcal{J}_t} \mid \{Z_i\}_{i\in\mathcal{I}_t}\big)\exp\left(\sum_{i\in\mathcal{J}_t}\Big(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2\Big)\right)(dz_j)_{j\in\mathcal{J}_t}}{\int_{\mathbb{R}^{|\mathcal{J}_t|}} \prod_{j\in\mathcal{J}_t} \mathbf{1}_{\{z_j > h_j(t)\}} p\big((z_j)_{j\in\mathcal{J}_t} \mid \{Z_i\}_{i\in\mathcal{I}_t}\big)\exp\left(\sum_{i\in\mathcal{J}_t}\Big(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2\Big)\right)(dz_j)_{j\in\mathcal{J}_t}},\end{aligned}$$

where $p\big((z_j)_{j\in\mathcal{J}_t} \mid \{Z_i\}_{i\in\mathcal{I}_t}\big)$ denotes the conditional joint density of $(Z_j)_{j\in\mathcal{J}_t}$ given the market factors for the obligors that defaulted up to and including time t. Additionally, the symbol $|A|$ stands for the number of elements in set A.

Proof. To begin with, we consider the case where only the first default happened up to time t. Based on our tentative rule about the rearrangement of the labels, we can divide $[n]$ into $\mathcal{J}_t = \{1, 2, \cdots, n-1\}$ and $\mathcal{I}_t = \{n\}$. Market participants can realize $Z_n = h_n(\tau_n)$ for $t \ge \tau_n$, and then the joint density of $(Z_j)_{j\in\mathcal{J}_t}$ would be altered at the first default time τ_n. Since $\tau_n \le t < \tau_j$ for $j \in \mathcal{J}_t$ and h_j is increasing, the market participants can recognize from (1) that $h_j(\tau_n) \le h_j(t) < h_j(\tau_j) = Z_j$ for $j \in \mathcal{J}_t$. Therefore, the first default time τ_n yields the information of $Z_n = h_n(\tau_n)$. Then we have, for $s \geq t \geq \tau_n$,

$$\begin{aligned}
&\mathbb{P}\left(\{\tau_\alpha > s\} \cap \{\tau_j > t \mid j \neq \alpha, n\}\Big|\ \{\xi_t^j\}_{j\in[n-1]},\ Z_n\right) \\
&= \mathbb{P}\left(\{Z_\alpha > h_\alpha(s)\} \cap \{Z_j > h_j(t) \mid j \neq \alpha, n\}\Big|\ \{\xi_t^j\}_{j\in[n-1]},\ \tau_n\right) \\
&= \int_{\mathbb{R}^{n-1}} \mathbb{P}\left(\{Z_\alpha > h_\alpha(s)\} \cap \{Z_j > h_j(t) \mid j \neq \alpha, n\}\Big|\ \{\xi_t^j\}_{j\in[n-1]},\ \tau_n, \{Z_j = z_j\}_{j\in[n-1]}\right) \\
&\qquad\qquad \times \mathbb{P}\big(\{Z_j \in dz_j\}_{j\in[n-1]} \mid \{\xi_t^j\}_{j\in[n-1]},\ \tau_n\big) \\
&= \int_{\mathbb{R}^{n-1}} \mathbf{1}_{\{z_\alpha > h_\alpha(s)\}} \prod_{j\neq\alpha,n} \mathbf{1}_{\{z_j > h_j(t)\}} \times \prod_{j\in[n-1]} \mathbf{1}_{\{z_j > h_j(t)\}} \mathbb{P}\big(\{Z_j \in dz_j\}_{j\in[n-1]} \mid \{\xi_t^j\}_{j\in[n-1]},\ \tau_n\big) \\
&= \int_{\mathbb{R}^{n-1}} \mathbf{1}_{\{z_\alpha > h_\alpha(s)\}} \prod_{j\neq\alpha,n} \mathbf{1}_{\{z_j > h_j(t)\}} \pi_t^{(n)}(dz_j)_{j\in[n-1]},
\end{aligned}$$

where we set

$$\pi_t^{(n)}(dz_j)_{j\in[n-1]} \ := \prod_{j\in[n-1]} \mathbf{1}_{\{z_j > h_j(t)\}} \mathbb{P}\big(\{Z_j \in dz_j\}_{j\in[n-1]} \mid \{\xi_t^j\}_{j\in[n-1]},\ Z_n\big)$$

as the conditional joint posterior of $(Z_j)_{j\in[n-1]}$ given $\{\xi_t^j\}_{j\in[n-1]}$ and τ_n. Using the Bayes formula,

$$\pi_t^{(n)}(dz_j)_{j\in[n-1]} = \frac{\displaystyle\prod_{j\in[n-1]} \mathbf{1}_{\{z_j > h_j(t)\}} p\big((z_j)_{j\in[n-1]} \mid Z_n\big) p\big((\xi_t^j)_{j\in[n-1]} \mid (z_j)_{j\in[n-1]}\big)(dz_j)_{j\in[n-1]}}{\displaystyle\int_{\mathbb{R}^{n-1}} \prod_{j\in[n-1]} \mathbf{1}_{\{z_j > h_j(t)\}} p\big((z_j)_{j\in[n-1]} \mid Z_n\big) p\big((\xi_t^j)_{j\in[n-1]} \mid (z_j)_{j\in[n-1]}\big)(dz_j)_{j\in[n-1]}}.$$

Here, the conditional normal distribution $p\big((z_j)_{j\in[n-1]} \mid Z_n\big)$ is obtained by its conditional mean vector and covariance matrix,

$$\mathbb{E}[(Z_1, \cdots, Z_{n-1})^\top | Z_n = z_n] = \Gamma_{12}\Gamma_{22}^{-1} z_n, \tag{7}$$

$$\mathrm{Cov}[(Z_1, \cdots, Z_{n-1})^\top | Z_n = z_n] = \Gamma_{11} - \Gamma_{12}\Gamma_{22}^{-1}\Gamma_{21}, \tag{8}$$

where the full covariance matrix $\Gamma \in \mathbb{R}^{n\times n}$ of $(Z_1, \cdots, Z_n)$ is assumed to be block structured as follows:

$$\Gamma = \left[\begin{array}{c|c} \Gamma_{11} & \Gamma_{12} \\ \hline \Gamma_{21} & \Gamma_{22} \end{array}\right],\ \Gamma_{11} \in \mathbb{R}^{(n-1)\times(n-1)}, \Gamma_{12} = \Gamma_{21}^\top \in \mathbb{R}^{(n-1)\times 1}, \Gamma_{22} \in \mathbb{R}.$$

Similar to Proposition 2.7, we remark that

$$\mathbb{P}\big((\xi_t^j)_{j\in[n-1]} \mid (z_j)_{j\in[n-1]}\big) = \frac{1}{(2\pi t)^{\frac{n-1}{2}}} \exp\left(-\frac{1}{2t}\Big[\sum_{j\in[n-1]} \big(\xi_t^j - \sigma_j t z_j\big)^2\Big]\right).$$

It follows

$$\begin{aligned}
&\mathbb{P}\left(\{\tau_\alpha > s\} \cap \{\tau_j > t \mid j \neq \alpha, n\} \Big| \ \{\xi_t^j\}_{j\in[n-1]},\ Z_n\right) \\
&= \frac{\displaystyle\int_{\mathbb{R}^{n-1}} \mathbf{1}_{\{z_\alpha > h_\alpha(s)\}} \left\{\prod_{j\neq\alpha,n} \mathbf{1}_{\{z_j > h_j(t)\}}\right\} p\big((z_j)_{j\in[n-1]} \mid Z_n\big) \exp\left(\sum_{i\in[n-1]} \Big(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2\Big)\right)(dz_j)_{j\in[n-1]}}{\displaystyle\int_{\mathbb{R}^{n-1}} \left\{\prod_{j\in[n-1]} \mathbf{1}_{\{z_j > h_j(t)\}}\right\} p\big((z_j)_{j\in[n-1]} \mid Z_n\big) \exp\left(\sum_{i\in[n-1]} \Big(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2\Big)\right)(dz_j)_{j\in[n-1]}}.
\end{aligned}$$

It remains to prove the general case (i.e., the default obligor set $\mathcal{I}_t$ has two or more elements). This can be achieved with a slight change in the above proof. Although it is possible to derive the formula recursively as time progresses, we can express it exactly by referring to the default history. Indeed, we have

$$\begin{aligned}
&\mathbb{P}\left(\{\tau_\alpha > s\} \cap \{\tau_j > t \mid j \in \mathcal{J}_t \backslash \{\alpha\}\} \Big| \ \{\xi_t^j\}_{j\in\mathcal{J}_t},\ \{Z_i\}_{i\in\mathcal{I}_t}\right) \\
&\quad = \int_{\mathbb{R}^{|\mathcal{J}_t|}} \mathbf{1}_{\{z_\alpha > h_\alpha(s)\}} \prod_{j\in\mathcal{J}_t\backslash\{\alpha\}} \mathbf{1}_{\{z_j > h_j(t)\}} \pi_t^{(\mathcal{I}_t)}(dz_j)_{j\in\mathcal{J}_t},
\end{aligned}$$

where the conditional joint posterior of $(Z_j)_{j\in\mathcal{J}_t}$ given $\{\xi_t^j\}_{j\in\mathcal{J}_t}$ and $\{Z_i\}_{i\in\mathcal{I}_t}$ is given as

$$\pi_t^{(\mathcal{I}_t)}(dz_j)_{j\in\mathcal{J}_t} := \prod_{i\in\mathcal{I}_t}\left(\prod_{j\in\mathcal{J}_t} \mathbf{1}_{\{z_j > h_j(t)\}}\right) \cdot \mathbb{P}\big(\{Z_j \in dz_j\}_{j\in\mathcal{J}_t} \big| \ \{\xi_t^j\}_{j\in\mathcal{J}_t}, \{Z_i = h_i(\tau_i)\}_{i\in\mathcal{I}_t}\big).$$

Hence, the Bayes formula implies that

$$\pi_t^{(\mathcal{I}_t)}(dz_j)_{j\in\mathcal{J}_t} = \frac{\displaystyle\prod_{j\in\mathcal{J}_t} \mathbf{1}_{\{z_j > h_j(t)\}} p\big((z_j)_{j\in\mathcal{J}_t} \mid \{Z_i\}_{i\in\mathcal{I}_t}\big) p\big((\xi_t^j)_{j\in\mathcal{J}_t} \mid (z_j)_{j\in\mathcal{J}_t}\big)(dz_j)_{j\in\mathcal{J}_t}}{\displaystyle\int_{-\infty}^{\infty} \prod_{j\in\mathcal{J}_t} \mathbf{1}_{\{z_j > h_j(t)\}} p\big((z_j)_{j\in\mathcal{J}_t} \mid \{Z_i\}_{i\in\mathcal{I}_t}\big) p\big((\xi_t^j)_{j\in\mathcal{J}_t} \mid (z_j)_{j\in\mathcal{J}_t}\big)(dz_j)_{j\in\mathcal{J}_t}}.$$

Therefore, it is straightforward to obtain the formula for the general case. Finally, we remark that the conditional joint distribution $p\big((z_j)_{j\in\mathcal{J}_t} \mid \{Z_i\}_{i\in\mathcal{I}_t}\big)$ can also be determined by the same formulas as (7) and (8) with the conditions $z_i = h_i(\tau_i)$ for all $i \in \mathcal{I}_t$. □

Remark 2.9. *It is a valuable information that "there has been no default up to time t." In this sense, Proposition 2.8 includes $\mathcal{I}_t = \emptyset$ as a special case if we interpret $\{Z_i\}_{i\in\emptyset}$ as the information of $Z_j > h_j(t)$ for any $j \in [n]$, and we substitute $p_0\left((z_j)_{j\in[n]}\right)$ for $p\left((z_j)_{j\in\mathcal{J}_t} \mid \{Z_i\}_{i\in\mathcal{I}_t}\right)$.*

Consequently, combining the generalized Dellacherie formula (4) with Propositions 2.7 and 2.8, we can derive the formula for the defaultable discount bond price. We note that the dynamics of $D_{t,T}^{(\alpha)}$ depend on ξ_t^α and ξ_t^i $(i \neq \alpha)$. This feature provides an enriched structure of interactions.

Corollary 2.10. *In the case of $n = 2$ with $Corr(Z_1, Z_2) = \rho \in (-1, 1)$, the defaultable discount bond price of obligor 1 is given by*

$$D_{t,T}^{(1)} = P_{t,T}\mathbf{1}_{\{\tau_1>t,\tau_2>t\}} \frac{\int_{h_1(T)}^{\infty}\int_{h_2(t)}^{\infty} p_0(z_1,z_2)\exp\Big(\sum_{i=1}^{2}(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2)\Big)dz_1 dz_2}{\int_{h_1(t)}^{\infty}\int_{h_2(t)}^{\infty} p_0(z_1,z_2)\exp\Big(\sum_{i=1}^{2}(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2)\Big)dz_1 dz_2} + P_{t,T}\mathbf{1}_{\{\tau_1>t,\tau_2\le t\}} \frac{\int_{h_1(T)}^{\infty} p(z_1 \mid Z_2)\exp\Big(\sigma_1 z_1 \xi_t^1 - \frac{t}{2}\sigma_1^2 z_1^2\Big)dz_1}{\int_{h_1(t)}^{\infty} p(z_1 \mid Z_2)\exp\Big(\sigma_1 z_1 \xi_t^1 - \frac{t}{2}\sigma_1^2 z_1^2\Big)dz_1}, \tag{9}$$

where

$$p_0(z_1,z_2) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\Big(-\frac{1}{2(1-\rho^2)}(z_1^2 - 2\rho z_1 z_2 + z_2^2)\Big),$$
$$p(z_1 \mid Z_2) = \frac{1}{\sqrt{2\pi(1-\rho^2)}}\exp\Big(-\frac{1}{2(1-\rho^2)}\big(z_1 - \rho\cdot Z_2\big)^2\Big).$$

Proof. We remember that the expression of $D_{t,T}^{(1)}$ seen in Example 2.6 contains the following conditional probabilities:

$$\mathbb{P}(\tau_1 > T, \tau_2 > t \mid \xi_t^1, \xi_t^2), \mathbb{P}(\tau_1 > t, \tau_2 > t \mid \xi_t^1, \xi_t^2), \mathbb{P}(\tau_1 > T \mid \xi_t^1, Z_2), \text{ and } \mathbb{P}(\tau_1 > t \mid \xi_t^1, Z_2).$$

We apply Proposition 2.7 with $n = 2$ and $\alpha = 1$ to obtain for $s \ge t$

$$\mathbb{P}(\tau_1 > s, \tau_2 > t \mid \xi_t^1, \xi_t^2) = \frac{\int_{h_1(s)}^{\infty}\int_{h_2(t)}^{\infty} p_0(z_1,z_2)\exp\Big(\sum_{i=1}^{2}(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2)\Big)dz_1 dz_2}{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_0(z_1,z_2)\exp\Big(\sum_{i=1}^{2}(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2)\Big)dz_1 dz_2}.$$

Similarly, it follows from Proposition 2.8 with $n = 2$ and $\alpha = 1, \mathcal{I}_t = \{2\}$ that for $s \ge t$

$$\mathbb{P}(\tau_1 > s \mid \xi_t^1, Z_2) = \frac{\int_{h_1(s)}^{\infty} p(z_1 \mid Z_2)\exp\Big(\sigma_1 z_1 \xi_t^1 - \frac{t}{2}\sigma_1^2 z_1^2\Big)dz_1}{\int_{h_1(h_2^{-1}(Z_2))}^{\infty} p(z_1 \mid Z_2)\exp\Big(\sigma_1 z_1 \xi_t^1 - \frac{t}{2}\sigma_1^2 z_1^2\Big)dz_1}.$$

Finally, the formula (9) can be obtained by substituting the expressions with $s = T$ or t in the form of a ratio of integrals as above for the conditional probabilities in Example 2.6. □

We note that the second term of (9) has a similar form for the single obligor case ($n = 1$) derived in Brody et al. (2010) if the conditional density $p(z_1 \mid \tau_2)$ is replaced by the unconditional density. Therefore, we are interested in how the first term of (9) works and what happens on the bond price issued by obligor 1 at the default time of obligor 2. To illustrate the default contagion effects of our model, we present some simulated trajectories of discount bond prices $\{D^{(1)}_{tT}\}_{0\le t\le T}$ using the Monte Carlo method based on Corollary 2.10.

We demonstrate a couple of cases: a highly correlated case $\rho = 0.8$ and a moderately correlated case $\rho = 0.4$. We suppose $r_t \equiv 0.05$(constant), $\sigma_1 = \sigma_2 = 1$, and $T = 1$ (year) for all the cases. For the numerical simulation, we discretize the time interval $[0, 1]$ into $0 = t_0, t_1, \cdots, t_{250} = T$ with a fixed time interval $\Delta t := t_k - t_{k-1} = 1/250$. In addition, we assume that the functions h_i $(i = 1, 2)$ are specified by $\tau_i = h_i^{-1}(Z_i) := -\log(\Phi(-Z_i))/\bar{\lambda}_i$ with parameters $\bar{\lambda}_1 = 0.02$ and $\bar{\lambda}_2 = 0.05$, respectively, where Φ denotes the standard normal distribution function. Such a specification of h_i follows from the naive assumption that the default time τ_i follows the exponential distribution with constant hazard rate $\bar{\lambda}_i$, namely, $\mathbb{P}(\tau_i > t) = \exp(-\bar{\lambda}_i t)$. To make it easier to see the contagion impact of obligor 2's default upon obligor 1, we assume that obligor 2 always defaults at $\tau_2 = 0.5$. This assumption implies that we fix $Z_2 = -1.9653$ from the definition of h_2 above. In addition, we set $Z_1 = -1.0$ so that obligor 1 never defaults during the interval $[0, 1]$. Thus, we fix the credit-related market factors as $(Z_1, Z_2) = (-1.0, -1.9653)$ for any case.

Figure 1 shows the three simulated sample trajectories on the interval [0,1] of the bond price process $D^{(1)}_{t,1}$ with fixed default time $\tau_2 = 0.5$ of obligor 2, respectively, for the case of $\rho = 0.8$ (left panel) and $\rho = 0.4$ (right panel). We can easily observe that the downward jump size of $D^{(1)}_{\tau_2,1}$ is larger for the highly correlated case $\rho = 0.8$ than for the moderately correlated case $\rho = 0.4$.

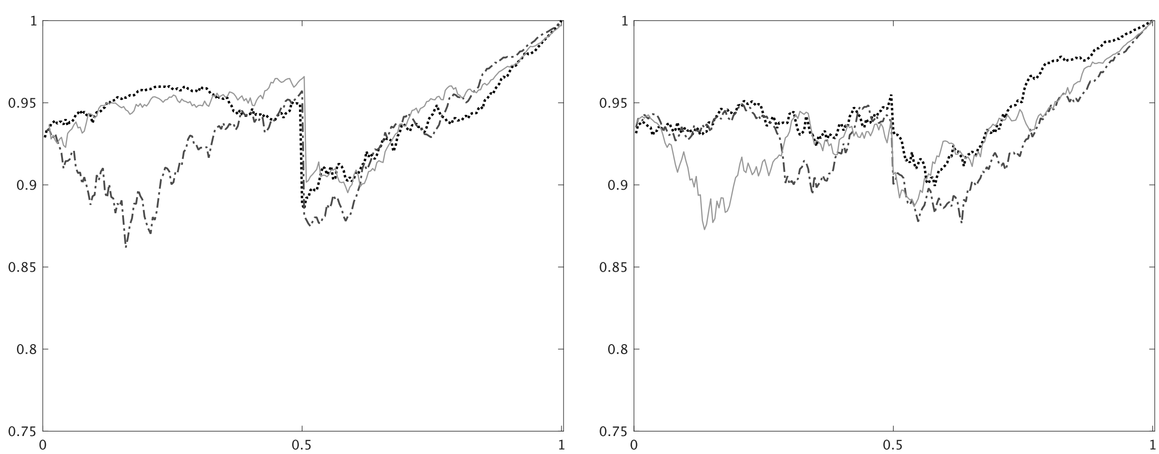

Figure 1. Simulated sample trajectories on the interval [0,1] of the bond price process $D^{(1)}_{t,1}$ with fixed default time $\tau_2 = 0.5$ of obligor 2. (Left panel) the case of $\rho = 0.8$. (Right panel) the case of $\rho = 0.4$

As we will see later, our formulation can be seen as a dynamical extension of the information-based default contagion in the factor Copula model described in subsection 9.8 in McNeil et al. (2005) to consider successive observation of noisy information associated with the factor vector. This extension enables us to see that the stochastic dynamics of bonds are mutually affected by one another in a more general form.

3. Main results for the case $n = 2$

This section aims to derive a system of stochastic differential equations that follow the defaultable discount bond price processes $\{D^{(i)}_{t,T}\}_{i\in[n]}$ given in (3) follow. Our main objective of this study is to investigate the default contagion impact on the active bond price processes in our model, so it is useful to see how these bonds interact in terms of stochastic differential equations. Here we show the result for the case of $n = 2$ to avoid complicated expressions for the general n case[1]. Appendix mentions the case of $n = 3$. We remember that the bond price processes $\{D^{(i)}_{t,T}\}_{i=1,2}$ are $\{\mathcal{G}_t\}$-adapted, so we expect that the bond prices processes can be represented in terms of some $\{\mathcal{G}_t\}$-Brownian motions derived from the continuous market information process $\{\xi^i_t\}_{i=1,2}$ as well as some $\{\mathcal{G}_t\}$-martingales associated with the jumps at default times $\{\tau_i\}_{i=1,2}$. Before the main theorem, we introduce $\{\mathcal{G}_t\}$-Brownian motions for the case of general n as follows.

Proposition 3.1 ($\{\mathcal{G}_t\}$-Brownian motions)**.** *Let*

$$W^{(i|\mathbb{G})}_t := \xi^i_t - \sigma_i \int_0^t \mathbb{E}[Z_i \mid \mathcal{G}_s] ds, \ i \in [n].$$

Then, $\{W^{(i|\mathbb{G})}_t\}_{i\in[n]}$ *are mutually independent* $\{\mathcal{G}_t\}$*-Brownian motions.*

Proof. As in the $n = 1$ case, which was proven in Brody et al. (2010), we rely on Levy's characterization theorem. See Theorem 3.6 of Revuz and Yor (1999). It follows from (2) that the bracket $\langle W^{(i|\mathbb{G})}, W^{(j|\mathbb{G})}\rangle_t$ can be calculated as follows.

$$\begin{aligned}\langle W^{(i|\mathbb{G})}, W^{(j|\mathbb{G})}\rangle_t &= \left\langle \xi^i - \sigma_i \int_0^\cdot \mathbb{E}[Z_i \mid \mathcal{G}_s] ds \ , \ \xi^i - \sigma_j \int_0^\cdot \mathbb{E}[Z_j \mid \mathcal{G}_s] ds \right\rangle_t \\ &= \langle \xi^i, \xi^j\rangle_t = \langle B^i, B^j\rangle_t = \delta_{ij} t.\end{aligned}$$

[1]This comes from the combinatorial explosion of the order of multiple defaults due to a *bottom up* approach. This difficulty would be mitigated if we consider homogeneous universe in the sense that the joint prior has homogeneous variances and pairwise correlations, and then simply focus on the order of defaults.

For $t \le u$ because $W_t^{(i|\mathbb{G})}$ is $\mathcal{G}_t$-measurable and $\{B_t^i\}$ is a $\{\mathcal{G}_t \vee \sigma(Z_i)\}$-Brownian motion, by using the tower property, it follows that

$$\begin{aligned}
\mathbb{E}\left[W_u^{(i|\mathbb{G})}\middle|\mathcal{G}_t\right] &= W_t^{(i|\mathbb{G})} + \mathbb{E}\left[\xi_u^i - \xi_t^i - \sigma_i \int_t^u \mathbb{E}[Z_i \mid \mathcal{G}_s]ds\middle|\mathcal{G}_t\right] \\
&= W_t^{(i|\mathbb{G})} + \mathbb{E}\left[\sigma_i Z_i(u-t) + B_u^i - B_t^i - \sigma_i \int_t^u \mathbb{E}[Z_i \mid \mathcal{G}_s]ds\middle|\mathcal{G}_t\right] \\
&= W_t^{(i|\mathbb{G})} + \mathbb{E}\left[\sigma_i Z_i(u-t) + \mathbb{E}\big[B_u^i - B_t^i\big|\mathcal{G}_t \vee \sigma(Z_i)\big] - \sigma_i \int_t^u \mathbb{E}[Z_i \mid \mathcal{G}_s]ds\middle|\mathcal{G}_t\right] \\
&= W_t^{(i|\mathbb{G})} + \sigma_i \mathbb{E}[Z_i \mid \mathcal{G}_t](u-t) - \sigma_i \int_t^u \mathbb{E}[\ \mathbb{E}[Z_i \mid \mathcal{G}_s] \mid \mathcal{G}_t]ds \\
&= W_t^{(i|\mathbb{G})} + \sigma_i \mathbb{E}[Z_i \mid \mathcal{G}_t](u-t) - \sigma_i \mathbb{E}[Z_i \mid \mathcal{G}_t](u-t) \\
&= W_t^{(i|\mathbb{G})}.
\end{aligned}$$

This implies that $\{W_t^{(i|\mathbb{G})}\}$ is a $\{\mathcal{G}_t\}$-martingale. Therefore, by Levy's characterizationt theorem, we can conclude that $\{W_t^{(i|\mathbb{G})} : i = 1, \cdots, n\}$ are mutually independent $\{\mathcal{G}_t\}$-Brownian motions. □

Remark 3.2. *We remark that the optional sampling theorem implies that $W_{t\wedge\tau_i}^{(i|\mathbb{G})}$ is a $\{\mathcal{G}_{t\wedge\tau_i}\}$-martingale, and thus a $\{\mathcal{G}_t\}$-martingale. In fact, we can see*

$$W_{t\wedge\tau_i}^{(i|\mathbb{G})} = \xi_{t\wedge\tau_i}^i - \sigma_i \int_0^{t\wedge\tau_i} \mathbb{E}[Z_i \mid \mathcal{G}_s]ds = \int_0^t \mathbf{1}_{\{\tau_i > s\}} \left(d\xi_s^i - \sigma_i \mathbb{E}[Z_i \mid \mathcal{G}_s]ds\right).$$

This representation implies that the process $W_{t\wedge\tau_i}^{(i|\mathbb{G})}$ is just the same as the martingale introduced by Brody et al. (2010).

Proposition 3.3 ($\{\mathcal{G}_t\}$-compensated jump martingales)**.** *Define the processes $\lambda_t^{(1|\mathbb{G})}$ and $\lambda_t^{(2|\mathbb{G})}$ by*

$$\begin{aligned}
\lambda_t^{(1|\mathbb{G})} &:= \mathbf{1}_{\{\tau_2 > t\}} \frac{\psi_{t,1}(h_1(t), h_2(t))}{\varphi_t(h_1(t), h_2(t); 1)} + \mathbf{1}_{\{\tau_2 \le t\}} \frac{\widehat{\psi}_{t,1}(h_1(t), h_2(\tau_2))}{\widehat{\varphi}_{t,1}(h_1(t), h_2(\tau_2); 1)}, \\
\lambda_t^{(2|\mathbb{G})} &:= \mathbf{1}_{\{\tau_1 > t\}} \frac{\psi_{t,2}(h_1(t), h_2(t))}{\varphi_t(h_1(t), h_2(t); 1)} + \mathbf{1}_{\{\tau_1 \le t\}} \frac{\widehat{\psi}_{t,2}(h_1(\tau_1), h_2(t))}{\widehat{\varphi}_{t,2}(h_1(\tau_1), h_2(t); 1)},
\end{aligned}$$

where for $z_1, z_2 \in \mathbb{R}$ and a random variable Y, we set

$$\begin{aligned}
\varphi_t(z_1, z_2; Y) &:= \mathbb{E}\left[\mathbf{1}_{\{Z_1 > z_1\}} \mathbf{1}_{\{Z_2 > z_2\}} Y \mid \xi_t^1, \xi_t^2\right], \\
\widehat{\varphi}_{t,1}(z_1, z_2; Y) &:= \mathbb{E}\left[\mathbf{1}_{\{Z_1 > z_1\}} Y \mid \xi_t^1, Z_2 = z_2\right], \quad \widehat{\varphi}_{t,2}(z_1, z_2; Y) := \mathbb{E}\left[\mathbf{1}_{\{Z_2 > z_2\}} Y \mid \xi_t^2, Z_1 = z_1\right] \\
\psi_{t,1}(z_1, z_2) &:= \mathbb{P}\left(Z_1 = z_1,\ Z_2 > z_2 \mid \xi_t^1, \xi_t^2\right), \quad \psi_{t,2}(z_1, z_2) := \mathbb{P}\left(Z_1 > z_1,\ Z_2 = z_2 \mid \xi_t^1, \xi_t^2\right), \\
\widehat{\psi}_{t,1}(z_1, z_2) &:= \mathbb{P}\left(Z_1 = z_1 \mid \xi_t^1, Z_2 = z_2\right), \quad \widehat{\psi}_{t,2}(z_1, z_2) := \mathbb{P}\left(Z_2 = z_2 \mid \xi_t^2, Z_1 = z_1\right).
\end{aligned}$$

Then the process $\lambda_t^{(1|\mathbb{G})}$ (resp. $\lambda_t^{(2|\mathbb{G})}$) is an $\{\mathcal{F}_t\}\vee\{\mathcal{H}_t^2\}$-default intensity of obligor 1 (resp. $\{\mathcal{F}_t\}\vee\{\mathcal{H}_t^1\}$-default intensity of obligor 2). In other words, the following compenseted jump processes $\mathbf{1}_{\{\tau_1\le t\}} - \int_0^t \mathbf{1}_{\{\tau_1>s\}}\lambda_s^{(1|\mathbb{G})}ds$ and $\mathbf{1}_{\{\tau_2\le t\}} - \int_0^t \mathbf{1}_{\{\tau_2>s\}}\lambda_s^{(2|\mathbb{G})}ds$ are $\{\mathcal{G}_t\}$-martingales.

Proof. We have

$$\begin{aligned}
\lambda_t^{(1|\mathbb{G})} &= \mathbf{1}_{\{\tau_2>t\}}\frac{\psi_{t,1}(h_1(t),h_2(t))}{\varphi_t(h_1(t),h_2(t);1)} + \mathbf{1}_{\{\tau_2\le t\}}\frac{\widehat{\psi}_{t,1}(h_1(t),h_2(\tau_2))}{\widehat{\varphi}_{t,1}(h_1(t),h_2(\tau_2);1)} \\
&= \mathbf{1}_{\{\tau_2>t\}}\frac{\mathbb{P}(Z_1=h_1(t),Z_2>h_2(t)\mid \xi_t^1,\xi_t^2)}{\mathbb{P}(Z_1>h_1(t),Z_2>h_2(t)\mid \xi_t^1,\xi_t^2)} + \mathbf{1}_{\{\tau_2\le t\}}\frac{\mathbb{P}(Z_1=h_1(t)\mid \xi_t^1,Z_2=h_2(\tau_2))}{\mathbb{P}(Z_1>h_1(t)\mid \xi_t^1,Z_2=h_2(\tau_2))} \\
&= \mathbf{1}_{\{\tau_2>t\}}\frac{\mathbb{P}(Z_1=h_1(t)\mid \xi_t^1,\xi_t^2,Z_2>h_2(t))}{\mathbb{P}(Z_1>h_1(t)\mid \xi_t^1,\xi_t^2,Z_2>h_2(t))} + \mathbf{1}_{\{\tau_2\le t\}}\frac{\mathbb{P}(Z_1=h_1(t)\mid \xi_t^1,Z_2=h_2(\tau_2))}{\mathbb{P}(Z_1>h_1(t)\mid \xi_t^1,Z_2=h_2(\tau_2))} \\
&= \mathbf{1}_{\{\tau_2>t\}}\frac{-\frac{\partial}{\partial t}\mathbb{P}(\tau_1>t\mid\mathcal{F}_t\vee\mathcal{H}_t^2)}{\mathbb{P}(\tau_1>t\mid\mathcal{F}_t\vee\mathcal{H}_t^2)} + \mathbf{1}_{\{\tau_2\le t\}}\frac{-\frac{\partial}{\partial t}\mathbb{P}(\tau_1>t\mid\mathcal{F}_t\vee\mathcal{H}_t^2)}{\mathbb{P}(\tau_1>t\mid\mathcal{F}_t\vee\mathcal{H}_t^2)} \\
&= \mathbf{1}_{\{\tau_2>t\}}\left(-\frac{\partial}{\partial t}\log\mathbb{P}(\tau_1>t\mid\mathcal{F}_t\vee\mathcal{H}_t^2)\right) + \mathbf{1}_{\{\tau_2\le t\}}\left(-\frac{\partial}{\partial t}\log\mathbb{P}(\tau_1>t\mid\mathcal{F}_t\vee\mathcal{H}_t^2)\right) \\
&= -\frac{\partial}{\partial t}\log\mathbb{P}(\tau_1>t\mid\mathcal{F}_t\vee\mathcal{H}_t^2),
\end{aligned}$$

which proves

$$\begin{aligned}
\exp\left(-\int_0^t \lambda_u^{(1|\mathbb{G})}du\right) &= \exp\left(-\int_0^t\Big(-\frac{\partial}{\partial u}\log\mathbb{P}(\tau_1>u\mid\mathcal{F}_u\vee\mathcal{H}_u^2)\Big)du\right) \\
&= \exp\Big(\log\mathbb{P}(\tau_1>t\mid\mathcal{F}_t\vee\mathcal{H}_t^2) - \log\mathbb{P}(\tau_1>0)\Big) \\
&= \mathbb{P}(\tau_1>t\mid\mathcal{F}_t\vee\mathcal{H}_t^2).
\end{aligned}$$

Consequently, $\lambda_t^{(1|\mathbb{G})}$ (resp. $\lambda_t^{(2|\mathbb{G})}$) can be seen as the instantaneous hazard rate process for τ_1 (resp. τ_2). Thus Proposition 5.1.3 in Bielecki and Rutkowski (2002) implies that $\lambda_t^{(1|\mathbb{G})}$ (resp. $\lambda_t^{(2|\mathbb{G})}$) can be regarded as the $\{\mathcal{F}_t\}\vee\{\mathcal{H}_t^2\}$-default intensity of obligor 1 (resp. $\{\mathcal{F}_t\}\vee\{\mathcal{H}_t^1\}$-default intensity of obligor 2), meaning that $\mathbf{1}_{\{\tau_1\le t\}} - \int_0^t \mathbf{1}_{\{\tau_1>s\}}\lambda_s^{(1|\mathbb{G})}ds$ (resp. $\mathbf{1}_{\{\tau_2\le t\}} - \int_0^t \mathbf{1}_{\{\tau_2>s\}}\lambda_s^{(2|\mathbb{G})}ds$) becomes a $\{\mathcal{G}_t\}$-martingale. □

Remark 3.4. *Strictly, the default intensity $\lambda_t^{(1|\mathbb{G})}$ (resp. $\lambda_t^{(2|\mathbb{G})}$) should be written by $\lambda_t^{(1|\mathbb{F}\vee\mathbb{H}^2)}$ (resp. $\lambda_t^{(2|\mathbb{F}\vee\mathbb{H}^1)}$) so as to clarify which filtration the process is adapted to. However, for notational simplicity, we use the notation $\lambda_t^{(1|\mathbb{G})}$ (resp. $\lambda_t^{(2|\mathbb{G})}$).*

From the last proposition it follows that our model can be viewed as a dynamic version of the so-called Kusuoka's counterexample model (c.f. Kusuoka (1999), Bielecki and Rutkowski (2002)) since the default intensities $\lambda_t^{(1|\mathbb{G})}$ and $\lambda_t^{(2|\mathbb{G})}$ are specified dependent on whether the counterpart has defaulted or not. Furthermore it follows from Example 2.6 that $\lambda_t^{(1|\mathbb{G})}$ can be regarded as the instantaneous credit spread at time t for obligor 1 on the set $\{\tau_1 > t\}$ as below:

$$
\begin{aligned}
&-\frac{\partial}{\partial T}\log\frac{D^{(1)}_{tT}}{P_{t,T}}\bigg|_{T=t} \\
&= \frac{P_{t,T}}{D^{(1)}_{tT}}\left\{\mathbf{1}_{\{\tau_2>t\}}\frac{-\frac{\partial}{\partial T}\mathbb{P}(\tau_1>T,\tau_2>t\mid\xi^1_t,\xi^2_t)}{\mathbb{P}(\tau_1>t,\tau_2>t\mid\xi^1_t,\xi^2_t)}+\mathbf{1}_{\{\tau_2\le t\}}\frac{-\frac{\partial}{\partial T}\mathbb{P}(\tau_1>T\mid\xi^1_t,Z_2)}{\mathbb{P}(\tau_1>t\mid\xi^1_t,Z_2)}\right\}\bigg|_{T=t} \\
&= \mathbf{1}_{\{\tau_2>t\}}\frac{\psi_{t,1}(h_1(t),h_2(t))}{\varphi_t(h_1(t),h_2(t);1)}+\mathbf{1}_{\{\tau_2\le t\}}\frac{\widehat{\psi}_{t,1}(h_1(t),h_2(\tau_2))}{\widehat{\varphi}_{t,1}(h_1(t),h_2(\tau_2);1)}=\lambda^{(1|\mathbb{G})}_t.
\end{aligned}
$$

Now, we state our main result on the stochastic differential equation followed by the bond price process $\{D^{(i)}_{tT}\}_{i=1,2}$ for the case of two debt obligors 1 and 2.

Theorem 3.5. *Let $W^{(i|\mathbb{G})}_t$ $(i=1,2)$ be the $\mathcal{G}_t\}$-Brownian motions defined in Proposition 3.1. Also, let $\lambda^{(i|\mathbb{G})}_t$ $(i=1,2)$ be the default intensities and $\varphi_t,\widehat{\varphi}_{t,i},\psi_{t,i},\widehat{\psi}_{t,i}$ $(i=1,2)$ be the functions, defined in Proposition 3.3. The defaultable discount bond price processes $\{D^{(1)}_{tT}\}$ and $\{D^{(2)}_{tT}\}$ with maturity T issued by obligor 1 and 2 respectively given in (9) satisfy the following two dimensional (backward) stochastic differential equation (SDE):*

$$
\begin{pmatrix} D^{(1)}_{TT} \\ D^{(2)}_{TT}\end{pmatrix}=\begin{pmatrix}\mathbf{1}_{\{\tau_1>T\}} \\ \mathbf{1}_{\{\tau_2>T\}}\end{pmatrix}
$$

and for $t<T$, we have

$$
\begin{aligned}
\begin{pmatrix} dD^{(1)}_{tT} \\ dD^{(2)}_{tT}\end{pmatrix} &= \begin{pmatrix} D^{(1)}_{t-,T} & 0 \\ 0 & D^{(2)}_{t-,T}\end{pmatrix}\left\{\begin{pmatrix} r_t+\lambda^{(1|\mathbb{G})}_t+\eta^{(2|\mathbb{G})}_{1:tT} \\ r_t+\lambda^{(2|\mathbb{G})}_t+\eta^{(1|\mathbb{G})}_{2:tT}\end{pmatrix}dt+\begin{pmatrix}\Sigma^{(1|\mathbb{G})}_{1:tT} & \Sigma^{(2|\mathbb{G})}_{1:tT} \\ \Sigma^{(1|\mathbb{G})}_{2:tT} & \Sigma^{(2|\mathbb{G})}_{2:tT}\end{pmatrix}\begin{pmatrix}\sigma_1 dW^{(1|\mathbb{G})}_t \\ \sigma_2 dW^{(2|\mathbb{G})}_t\end{pmatrix}\right. \\
&\qquad\left. -\begin{pmatrix} 1 & 1-\Xi^{(2|\mathbb{G})}_{1:tT} \\ 1-\Xi^{(1|\mathbb{G})}_{2:tT} & 1\end{pmatrix}\begin{pmatrix} d\mathbf{1}_{\{\tau_1\le t\}} \\ d\mathbf{1}_{\{\tau_2\le t\}}\end{pmatrix}\right\} \\
&= \begin{pmatrix} D^{(1)}_{t-,T} & 0 \\ 0 & D^{(2)}_{t-,T}\end{pmatrix}\left\{r_t dt+\begin{pmatrix}\Sigma^{(1|\mathbb{G})}_{1:tT} & \Sigma^{(2|\mathbb{G})}_{1:tT} \\ \Sigma^{(1|\mathbb{G})}_{2:tT} & \Sigma^{(2|\mathbb{G})}_{2:tT}\end{pmatrix}\begin{pmatrix}\sigma_1 dW^{(1|\mathbb{G})}_t \\ \sigma_2 dW^{(2|\mathbb{G})}_t\end{pmatrix}\right. \\
&\qquad\left. -\begin{pmatrix} 1 & 1-\Xi^{(2|\mathbb{G})}_{1:tT} \\ 1-\Xi^{(1|\mathbb{G})}_{2:tT} & 1\end{pmatrix}\begin{pmatrix} d\mathbf{1}_{\{\tau_1\le t\}}-\mathbf{1}_{\{\tau_1>t\}}\lambda^{(1|\mathbb{G})}_t \\ d\mathbf{1}_{\{\tau_2\le t\}}-\mathbf{1}_{\{\tau_2>t\}}\lambda^{(2|\mathbb{G})}_t\end{pmatrix}\right\},
\end{aligned}
$$

where

$$\Xi^{(2|\mathbb{G})}_{1:tT} := \frac{\varphi_t(h_1(t),h_2(t);1)}{\varphi_t(h_1(T),h_2(t);1)}\frac{\widehat{\varphi}_{t,1}(h_1(T),h_2(t);1)}{\widehat{\varphi}_{t,1}(h_1(t),h_2(t);1)}, \quad \Xi^{(1|\mathbb{G})}_{2:tT} := \frac{\varphi_t(h_1(t),h_2(t);1)}{\varphi_t(h_1(t),h_2(T);1)}\frac{\widehat{\varphi}_{t,2}(h_1(t),h_2(T);1)}{\widehat{\varphi}_{t,2}(h_1(t),h_2(t);1)},$$

$$\eta^{(2|\mathbb{G})}_{1:tT} := \mathbf{1}_{\{\tau_2>t\}}\Bigg(\underbrace{\frac{\psi_{t,2}(h_1(t),h_2(t))}{\varphi_t(h_1(t),h_2(t);1)}}_{=\mathbf{1}_{\{\tau_1>t\}}\lambda^{(2|\mathbb{G})}_t} - \frac{\psi_{t,2}(h_1(T),h_2(t))}{\varphi_t(h_1(T),h_2(t);1)}\Bigg)\left(= \mathbf{1}_{\{\tau_2>t\}}\lambda^{(2|\mathbb{G})}_t\left(1-\Xi^{(2|\mathbb{G})}_{1:tT}\right) \ \textit{on}\ \{\tau_1>t\}\right),$$

$$\eta^{(1|\mathbb{G})}_{2:tT} := \mathbf{1}_{\{\tau_1>t\}}\Bigg(\underbrace{\frac{\psi_{t,1}(h_1(t),h_2(t))}{\varphi_t(h_1(t),h_2(t);1)}}_{=\mathbf{1}_{\{\tau_2>t\}}\lambda^{(1|\mathbb{G})}_t} - \frac{\psi_{t,1}(h_1(t),h_2(T))}{\varphi_t(h_1(t),h_2(T);1)}\Bigg)\left(= \mathbf{1}_{\{\tau_1>t\}}\lambda^{(1|\mathbb{G})}_t\left(1-\Xi^{(1|\mathbb{G})}_{2:tT}\right) \ \textit{on}\ \{\tau_2>t\}\right),$$

$$\begin{aligned}\Sigma^{(1|\mathbb{G})}_{1:tT} &:= \mathbf{1}_{\{\tau_2>t\}}\left(\frac{\varphi_t(h_1(T),h_2(t);Z_1)}{\varphi_t(h_1(T),h_2(t);1)} - \frac{\varphi_t(h_1(t),h_2(t);Z_1)}{\varphi_t(h_1(t),h_2(t);1)}\right)\\ &\quad + \mathbf{1}_{\{\tau_2\le t\}}\left(\frac{\widehat{\varphi}_{t,1}(h_1(T),h_2(\tau_2);Z_1)}{\widehat{\varphi}_{t,1}(h_1(T),h_2(\tau_2);1)} - \frac{\widehat{\varphi}_{t,1}(h_1(t),h_2(\tau_2);Z_1)}{\widehat{\varphi}_{t,1}(h_1(t),h_2(\tau_2);1)}\right),\\ \Sigma^{(2|\mathbb{G})}_{1:tT} &:= \mathbf{1}_{\{\tau_2>t\}}\left(\frac{\varphi_t(h_1(T),h_2(t);Z_2)}{\varphi_t(h_1(T),h_2(t);1)} - \frac{\varphi_t(h_1(t),h_2(t);Z_2)}{\varphi_t(h_1(t),h_2(t);1)}\right),\end{aligned}$$

$$\begin{aligned}\Sigma^{(1|\mathbb{G})}_{2:tT} &:= \mathbf{1}_{\{\tau_1>t\}}\left(\frac{\varphi_t(h_1(t),h_2(T);Z_1)}{\varphi_t(h_1(t),h_2(T);1)} - \frac{\varphi_t(h_1(t),h_2(t);Z_1)}{\varphi_t(h_1(t),h_2(t);1)}\right),\\ \Sigma^{(2|\mathbb{G})}_{2:tT} &:= \mathbf{1}_{\{\tau_1>t\}}\left(\frac{\varphi_t(h_1(t),h_2(T);Z_2)}{\varphi_t(h_1(t),h_2(T);1)} - \frac{\varphi_t(h_1(t),h_2(t);Z_2)}{\varphi_t(h_1(t),h_2(t);1)}\right)\\ &\quad + \mathbf{1}_{\{\tau_1\le t\}}\left(\frac{\widehat{\varphi}_{t,2}(h_1(\tau_1),h_2(T);Z_2)}{\widehat{\varphi}_{t,2}(h_1(\tau_1),h_2(T);1)} - \frac{\widehat{\varphi}_{t,2}(h_1(\tau_1),h_2(t);Z_2)}{\widehat{\varphi}_{t,2}(h_1(\tau_1),h_2(t);1)}\right).\end{aligned}$$

As for the functions φ_t and ψ, we can enlarge their definition to the case of general n. Specifically, for any sets $\mathcal{I},\mathcal{J}\subset[n]$ with $\mathcal{I}\cup\mathcal{J}=[n]$, $\mathcal{I}\cap\mathcal{J}=\emptyset$, we define as below. For $z_1,\cdots,z_n\in\mathbb{R}$ and a random variable Y, we define

$$\varphi_{t,\mathcal{J}}((z_j)_{j\in[n]};Y) := \mathbb{E}\left[\prod_{j\in\mathcal{J}}\mathbf{1}_{\{Z_j>z_j\}}Y \mid (\xi^j_t)_{j\in\mathcal{J}},\{Z_i=z_i\}_{i\in\mathcal{I}}\right], \tag{10}$$

$$\psi_{t,k,\mathcal{J}}((z_j)_{j\in[n]}) := \mathbb{P}\left(Z_k=z_k,\{Z_j>z_j\}_{j\in\mathcal{J}\setminus\{k\}} \mid (\xi^j_t)_{j\in\mathcal{J}},\{Z_i=z_i\}_{i\in\mathcal{I}}\right) \quad \text{for } k\in\mathcal{J}. \tag{11}$$

In this sense, we note that the notation for $n=2$ in the theorem is redefined for simplicity as follows:

$$\varphi_t(z_1,z_2;Y)=\varphi_{t,[2]}(z_1,z_2;Y), \quad \widehat{\varphi}_t(z_1,z_2;Y)=\varphi_{t,\{1\}}(z_1,z_2;Y),$$
$$\psi_{t,k}(z_1,z_2)=\psi_{t,k,[2]}\ (k=1,2), \quad \widehat{\psi}_{t,1}(z_1,z_2)=\psi_{t,1,\{1\}}(z_1,z_2).$$

There are some considerations on the stochastic differential equations in the theorem. Because of symmetry, we discuss only from the perspective of obligor 1 hereafter. Hence, for notational convenience, we will write $\eta_{tT}^{(2|\mathbb{G})}$ (resp. $\Xi_{tT}^{(2|\mathbb{G})}, \Sigma_{t,T}^{(1|\mathbb{G})}, \Sigma_{t,T}^{(2|\mathbb{G})}$) for $\eta_{1:tT}^{(2|\mathbb{G})}$ (resp. $\Xi_{1:tT}^{(2|\mathbb{G})}, \Sigma_{1:t,T}^{(1|\mathbb{G})}, \Sigma_{1:t,T}^{(2|\mathbb{G})}$) when it is clear from the context that we are discussing about the obligor 1.

First, we notice that all the stochastic drivers of the bond price process can be regarded as $\{\mathcal{G}_t\}$-martingales since the discounted bond price process $\{P_{tT}^{-1} D_{tT}^{(1)}\}_{t\in[0,T]}$ is a $\{\mathcal{G}_t\}$-martingale. Indeed, we find that the second representation implies that the dynamics of the defaultable discount bond $D_{tT}^{(1)}$ before the counterpart obligor 2's default time τ_2 is driven by the $\{\mathcal{G}_t\}$-Brownian motions $(W_t^{(1|\mathbb{G})}, W_t^{(2|\mathbb{G})})$ as well as the compensated default indicator processes, $\mathbf{1}_{\{\tau_1\le t\}} - \int_0^t \mathbf{1}_{\{\tau_1>s\}}\lambda_s^{(1|\mathbb{G})}ds$ and $\mathbf{1}_{\{\tau_2\le t\}} - \int_0^t \mathbf{1}_{\{\tau_2>s\}}\lambda_s^{(2|\mathbb{G})}ds$. On the other hand, after obligor 2's default, the defaultable discount bond $D_{tT}^{(1)}$ is driven only by $W_t^{(1|\mathbb{G})}$ and $\mathbf{1}_{\{\tau_1\le t\}} - \int_0^t \mathbf{1}_{\{\tau_1>s\}}\lambda_s^{(1|\mathbb{G})}ds$, and the default intensity $\lambda_t^{(1|\mathbb{G})}$ and the volatility $\Sigma_{1:tT}^{(1|\mathbb{G})}$ for obligor 1 are different from the ones before τ_2.

Second, we note that the jump impact of obligor 2's default at time τ_2 on the bond price $D_{tT}^{(1)}$ is given as follows:

$$
\begin{aligned}
D_{\tau_2,T}^{(1)} - D_{\tau_2-,T}^{(1)} &= -D_{\tau_2-,T}^{(1)}\left(1 - \Xi_{\tau_2,T}^{(2|\mathbb{G})}\right) \\
&= P_{\tau_2 T}\left(\frac{\widehat{\varphi}_{\tau_2}(h_1(T), h_2(\tau_2); 1)}{\widehat{\varphi}_{\tau_2}(h_1(\tau_2), h_2(\tau_2); 1)} - \frac{\varphi_{\tau_2}(h_1(T), h_2(\tau_2); 1)}{\varphi_{\tau_2}(h_1(\tau_2), h_2(\tau_2); 1)}\right).
\end{aligned}
$$

The last equality follows from the argument in Subsection 4.3. This implies that the bond price can jump at the default time of the counterpart obligor because the information is largely updated by revealing the market factor of the counterpart, even though the bond does not default due to the counterpart obligor's default. Therefore, we can regard $\Xi_{tT}^{(2|\mathbb{G})}$ as the "pseudo recovery rate" (or $1 - \Xi_{tT}^{(2|\mathbb{G})}$ stands for the "pseudo loss rate") by obligor 2's default since it looks like the recovery rate of market value despite not actually falling into default. This consideration means that the obligor 1's bond is faced with the risk which falls into pseudo default due to the obligor 2's default.

Third, we observe the equality $\eta_{tT}^{(2|\mathbb{G})} = \lambda_t^{(2|\mathbb{G})}(1 - \Xi_{tT}^{(2|\mathbb{G})})$. While we can interpret $\eta_{tT}^{(2|\emptyset)}$ as the difference of the conditional hazard rates for the obligor 2 given obligor 1's survival between time t and T, we view $\lambda_t^{(2|\mathbb{G})}(1 - \Xi_{tT}^{(2|\mathbb{G})})$ as the product of the instantaneous hazard rate of obligor 2 and "pseudo loss rate" by obligor 2's default as one can see from the previous two considerations. Such a specification is similar to the argument that the credit spread can be specified by the hazard rate and the fractional recovery (or loss given default) of market value (c.f. Duffie and Singleton (1999), Bielecki and Rutkowski (2002)). In this sense, it seems interesting that the trend term in the first SDE representation implies that the excess rate over the default-free interest rate r_t is composed of not only the term $\lambda_t^{(1|\mathbb{G})}$, the instantaneous hazard rate or credit spread of obligor 1, but also the term $\mathbf{1}_{\{\tau_2>t\}}\eta_{tT}^{(2|\mathbb{G})}$ regarding the credit quality of obligor 2, although obligor 2's default does not necessarily cause the default of the bond issued by obligor 1. Also, we discuss the sign of the term $\eta_{tT}^{(2|\mathbb{G})}$ in the proposition below. As is shown in Proposition 3.6 below, if the market factors are negatively correlated, the component $\eta_{tT}^{(2|\mathbb{G})}$ can be negative; hence, the trend term can shrink compared to when the underlying bond is evaluated alone. In other words, whether the bond price jumps upward or downward depends on the sign of the correlation parameter ρ between both market factors. The impact of defaults on the price of defaultable securities are discussed in some previous studies: a copula dependent model (McNeil et al. (2005)), an information-based default contagion model (Section 9.8 of McNeil et al. (2005)), and a density approach (El Karoui et al. (2015), Crépey et al. (2013), Crépey and Song (2017)).

Finally, we mention that the volatility component $\Sigma_{tT}^{(j|\mathbb{G})}$, corresponding to the Brownian motion term $dW_t^{(j|\mathbb{G})}$, can be regarded as the difference between the following conditional expectations: for $\tau_2 > t$,

$$\Sigma_{tT}^{(j|\mathbb{G})} = \mathbb{E}\left[Z_j \mid \mathcal{F}_t, Z_1 > h_1(T), Z_2 > h_2(t)\right] - \mathbb{E}\left[Z_j \mid \mathcal{F}_t, Z_1 > h_1(t), Z_2 > h_2(t)\right] \quad (j = 1, 2)$$

or for $\tau_2 \leq t$

$$\Sigma_{tT}^{(1|\mathbb{G})} = \mathbb{E}\left[Z_1 \mid \mathcal{F}_t, Z_1 > h_1(T), Z_2\right] - \mathbb{E}\left[Z_1 \mid \mathcal{F}_t, Z_1 > h_1(t), Z_2\right], \quad \text{and } \Sigma_{tT}^{(2|\mathbb{G})} = 0.$$

As the function h_1 is increasing, we have $h_1(T) > h_1(t)$ for $T > t$. Thus we have $\Sigma_{tT}^{(j|\mathbb{G})} > 0$ for $T > t$.

Here, in connection with the above considerations, we mention the signs of some processes.

Proposition 3.6. *We have the following properties:*

(i) For $T > t$, $\Sigma_{k:tT}^{(i|\mathbb{G})} > 0$ a.s. for $i, k = 1, 2$.
(ii) If $\rho > 0$ (resp. $\rho < 0$), then $\eta_{tT}^{(i|\mathbb{G})} > 0$ (resp. $\eta_{tT}^{(i|\mathbb{G})} < 0$) a.s. for $i = 1, 2$.

Proof. (i) It is proved for the case of $i = 1$ in the above discussion. The case of $i = 2$ can be proved by a similar argument.

(ii) We prove only the case of $i = 1$ under the condition $\{\tau_1 > t, \tau_2 > t\}$. Setting

$$A(s) := \frac{\psi_{t,2}(h_1(s), h_2(t))}{\varphi_t(h_1(s), h_2(t); 1)} \quad s \geq t$$

leads to $\eta_{tT}^{(2|\mathbb{G})} = A(t) - A(T)$ fot $t < T$. Then we see that $\eta_{tT}^{(2|\mathbb{G})} > 0$ is equivalent to the condition that $A(s)$ decreases with respect to s. The assertion is shown by calculating $\frac{\partial}{\partial s}A(s)$ directly for fixed ξ^1 and ξ^2 to verify that $\frac{\partial}{\partial s}A(s)$ is dependent on the sign of ρ. □

We finish this section with a remark about the existence and uniqueness of solution of linear BSDE with jumps like the one that appeared in Theorem 3.5.

Remark 3.7. *The existence and uniqueness of solution of linear BSDEs with jumps is discussed in Quenez and Sulem (2013), for instance. In standard expression of the BSDE theory, the equation which we achieve as the scalar-valued form (23) in Section 4 can be represented by* $D^{(1)}_{TT} = \mathbf{1}_{\{\tau_1>T\}}$ *and*

$$\begin{aligned}-dD^{(1)}_{tT} &= -r_t D^{(1)}_{t-,T}dt - D^{(1)}_{t-,T}\left\{\sigma_1\Sigma^{(1|\mathbb{G})}_{tT}dW^{(1|\mathbb{G})}_t + \sigma_2\Sigma^{(2|\mathbb{G})}_{tT}dW^{(2|\mathbb{G})}_t\right\}\\ &\quad - D^{(1)}_{t-,T}\left\{-\left(d\mathbf{1}_{\{\tau_1\le t\}} - \mathbf{1}_{\{\tau_1>t\}}\lambda^{(1|\mathbb{G})}_t dt\right) - \left(1-\Xi^{(2|\mathbb{G})}_{tT}\right)\left(d\mathbf{1}_{\{\tau_2\le t\}} - \mathbf{1}_{\{\tau_2>t\}}\lambda^{(2|\mathbb{G})}_t dt\right)\right\}.\end{aligned}$$

Strictly speaking, the solution of the BSDE should be given by a triplet of the defaultable discount bond price process, the predictable processes of coefficient with respect to Brownian motions and the compensated point processes, namely,

$$\left(D^{(1)}_{tT},\ \left(D^{(1)}_{t-,T}\sigma_1\Sigma^{(1|\mathbb{G})}_{t-,T}, D^{(1)}_{t-,T}\sigma_2\Sigma^{(2|\mathbb{G})}_{t-,T}\right),\ \left(-D^{(1)}_{t-,T}, -D^{(1)}_{t-,T}\left(1-\Xi^{(2|\mathbb{G})}_{tT}\right)\right) \right),$$

where $D^{(1)}_{tT}$ *satisfies* $P_{t,T}\mathbb{E}[\mathbf{1}_{\{\tau_\alpha>T\}}|\mathcal{G}_t]$. *Note that the second component is a predictable version of the processes* $(D^{(1)}_{t-,T}\sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}, D^{(1)}_{t-,T}\sigma_2\Sigma^{(2|\mathbb{G})}_{1:tT})$. *In fact, this can be regarded as the unique solution of the BSDE by applying a slight extension of the results in sections 2 and 3 of Quenez and Sulem (2013) to the above BSDE, since the conditions on regularity and measurability of the processes appeared in the above BSDE are consistent with the argument on linear BSDEs with jumps in Quenez and Sulem (2013).*

4. Proof of main Theorem

In this section, we prove Theorem 3.5. Due to the symmetry between the two bonds, we focus on only the process $\{D^{(1)}_{tT}\}$. For this end, we refer to the representation of $D^{(1)}_{tT}$ given by (9) in Corollary 2.10 and introduce two families of $\{\mathcal{G}_t\}$-adapted continuous processes parameterized by u_i as follows.

$$F^{(1|\emptyset)}_{t,u_1u_2} := \int_{h_1(u_1)}^{\infty}\int_{h_2(u_2)}^{\infty} p_0(z_1,z_2)\exp\left(\sum_{i=1}^{2}\left(\sigma_i z_i\xi^i_t - \frac{1}{2}\sigma_i^2 z_i^2 t\right)\right)dz_1dz_2 = \varphi_t(h_1(u_1),h_2(u_2);1), \tag{12}$$

$$F^{(1|2)}_{t,u_1} := \int_{h_1(u_1)}^{\infty} p\big(z_1 \mid h_2(\tau_2)\big)\exp\Big(\sigma_1 z_1\xi^1_t - \frac{1}{2}\sigma_1^2 z_1^2 t\Big)dz_1 = \widehat{\varphi}_t(h_1(u_1),h_2(\tau_2);1), \quad \tau_2\le t. \tag{13}$$

Then, from (9) in Corollary 2.10, $D^{(1)}_{tT}$ can be represented as

$$D^{(1)}_{tT} \ = \ D^{(1)}_{tT}\mathbf{1}_{\{\tau_1>t\}} = P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}\frac{F^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,tt}} \ + \ P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}. \tag{14}$$

4.1. **The dynamics of $\mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}D^{(1)}_{tT}$.** We first examine the second term in (14). From the integration by-parts formula for the product of the three processes $P_{tT}, \mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}$ and $\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}$, and the fact that all the bracket terms vanish, it follows that

$$\begin{aligned} d\Big(P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}\Big) &= r_tP_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}dt + P_{tT}\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}d\mathbf{1}_{\{\tau_1>t,\tau_2\le t\}} \\ &\quad + P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}d\left(\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}\right) \\ &= r_tD^{(1)}_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}dt + P_{tT}\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}\left(-\mathbf{1}_{\{\tau_2<t\}}d\mathbf{1}_{\{\tau_1\le t\}} + \mathbf{1}_{\{\tau_1\ge t\}}d\mathbf{1}_{\{\tau_2\le t\}}\right) \\ &\quad + P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}d\left(\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}\right). \end{aligned}$$

The term $d\left(\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}\right)$ can be seen as the stochastic differentiated form of the quotient of $F^{(1|2)}_{t,T}$ and $F^{(1|2)}_{t,t}$, so the Ito formula implies that

$$\begin{aligned} &P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}d\left(\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}\right) \\ &\quad = \mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}D^{(1)}_{tT}\left[\frac{dF^{(1|2)}_{t,T}}{F^{(1|2)}_{t,T}} - \frac{dF^{(1|2)}_{t,t}}{F^{(1|2)}_{t,t}} + \left(\frac{dF^{(1|2)}_{t,t}}{F^{(1|2)}_{t,t}}\right)^2 - \frac{d\langle F^{(1|2)}_{\cdot,T}, F^{(1|2)}_{\cdot,t}\rangle_t}{F^{(1|2)}_{t,T}F^{(1|2)}_{t,t}}\right], \end{aligned} \tag{15}$$

which motivates us to calculate the stochastic differential of (13) for $u_1 = T$ and $u_1 = t$. It is easy to see that

$$d\left(\exp\left(\sigma z\xi_t - \frac{1}{2}\sigma^2z^2t\right)\right) = \exp\left(\sigma z\xi_t - \frac{1}{2}\sigma^2z^2t\right)\sigma zd\xi_t,$$

therefore, we have

$$\begin{aligned} dF^{(1|2)}_{t,T} &= \int_{h_1(T)}^{\infty} p(z_1\mid h_2(\tau_2))\left(e^{\sigma_1z_1\xi^1_t-\frac{1}{2}\sigma_1^2z_1^2t}\right)\sigma_1z_1d\xi^1_tdz_1 \\ &= \frac{\sigma_1\displaystyle\int_{h_1(\tau_2)}^{\infty} p(z_1\mid h_2(\tau_2))\mathbf{1}_{\{z_1>h_1(T)\}}z_1e^{\sigma_1z_1\xi^1_t-\frac{1}{2}\sigma_1^2z_1^2t}d\xi^1_tdz_1}{\displaystyle\int_{h_1(\tau_2)}^{\infty} p(z_1\mid h_2(\tau_2))e^{\sigma_1z_1\xi^1_t-\frac{1}{2}\sigma_1^2z_1^2t}dz_1}\int_{h_1(\tau_2)}^{\infty} p(z_1\mid h_2(\tau_2))e^{\sigma_1z_1\xi^1_t-\frac{1}{2}\sigma_1^2z_1^2t}dz_1 \\ &= \sigma_1\mathbb{E}\Big[\mathbf{1}_{\{Z_1>h_1(T)\}}Z_1\Big|\xi^1_t, h_2(\tau_2)\Big]d\xi^1_t \;\times\; \int_{h_1(\tau_2)}^{\infty} p(z_1\mid h_2(\tau_2))e^{\sigma_1z_1\xi^1_t-\frac{1}{2}\sigma_1^2z_1^2t}dz_1. \end{aligned}$$

The last equality is valid since it follows from the argument in the proof of Proposition 2.8 that

$$\mathbb{P}(Z_1\in dz_1\mid \xi^1_t, h_2(\tau_2)) = \frac{\mathbf{1}_{\{z_1>h_1(\tau_2)\}}p(z_1\mid h_2(\tau_2))\left(e^{\sigma_1z_1\xi^1_t-\frac{1}{2}\sigma_1^2z_1^2t}\right)dz_1}{\displaystyle\int_{h_1(\tau_2)}^{\infty} p(z_1\mid h_2(\tau_2))e^{\sigma_1z_1\xi^1_t-\frac{1}{2}\sigma_1^2z_1^2t}dz_1}.$$

Then, we have

$$\frac{dF^{(1|2)}_{t,T}}{F^{(1|2)}_{t,T}} = \sigma_1 \frac{\widehat{\varphi}_t(h_1(T), h_2(\tau_2); Z_1)}{\widehat{\varphi}_t(h_1(T), h_2(\tau_2); 1)} d\xi^1_t. \tag{16}$$

On the contrary, we have

$$\begin{aligned}
dF^{(1|2)}_{t,t} &= \int_{-\infty}^{\infty} p(z_1 \mid h_2(\tau_2)) d\Big(\mathbf{1}_{\{h_1^{-1}(z_1)>t\}} e^{\sigma_1 z_1 \xi^1_t - \frac{1}{2}\sigma_1^2 z_1^2 t}\Big) dz_1 \\
&= \left(-\mathbb{E}\Big[\delta_{\{h_1^{-1}(Z_1)-t\}} \mid \xi^1_t, h_2(\tau_2)\Big] dt \ + \ \sigma_1 \mathbb{E}\Big[\mathbf{1}_{\{h_1^{-1}(Z_1)>t\}} Z_1 \mid \xi^1_t, h_2(\tau_2)\Big] d\xi^1_t \right) \\
&\qquad\qquad \times \int_{h_1(h_2^{-1}(Z_2))}^{\infty} p(z_1 \mid h_2(\tau_2)) e^{\sigma_1 z_1 \xi^1_t - \frac{1}{2}\sigma_1^2 z_1^2 t} dz_1.
\end{aligned}$$

Hence, we obtain

$$\frac{dF^{(1|2)}_{t,t}}{F^{(1|2)}_{t,t}} = - \underbrace{\frac{\widehat{\psi}_{t,1}(h_1(t), h_2(\tau_2))}{\widehat{\varphi}_t(h_1(t), h_2(\tau_2); 1)}}_{= \mathbf{1}_{\{\tau_2 < t\}} \lambda^{(1|\mathbb{G})}_t} dt + \sigma_1 \frac{\widehat{\varphi}_t(h_1(t), h_2(\tau_2); Z_1)}{\widehat{\varphi}_t(h_1(t), h_2(\tau_2); 1)} d\xi^1_t. \tag{17}$$

Substituting (16) and (17) for (15) and using $d\langle \xi^1, \xi^1 \rangle_t = dt$ from definition (2), we obtain

$$\begin{aligned}
& P_{tT} \mathbf{1}_{\{\tau_1 > t, \tau_2 \le t\}} d\Big(\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}\Big) \\
&= \mathbf{1}_{\{\tau_1 > t, \tau_2 \le t\}} D^{(1)}_{tT} \left\{ \frac{\widehat{\psi}(h_1(t), h_2(\tau_2))}{\widehat{\varphi}_t(h_1(t), h_2(\tau_2); 1)} dt \right. \\
&\quad \left. + \sigma_1 \underbrace{\left(\frac{\widehat{\varphi}_t(h_1(T), h_2(\tau_2); Z_1)}{\widehat{\varphi}_t(h_1(T), h_2(\tau_2); 1)} - \frac{\widehat{\varphi}_t(h_1(t), h_2(\tau_2); Z_1)}{\widehat{\varphi}_t(h_1(t), h_2(\tau_2); 1)} \right)}_{= \Sigma^{(1|\mathbb{G})}_{1:tT}} \left(d\xi^1_t - \sigma_1 \frac{\widehat{\varphi}_t(h_1(t), h_2(\tau_2); Z_1)}{\widehat{\varphi}_t(h_1(t), h_2(\tau_2); 1)} dt \right) \right\}.
\end{aligned}$$

From Proposition 2.5, the Markov property of $\{\xi^1_t\}$, $\sigma(Z_2) = \sigma(\tau_2)$ and the property that the event $\{\tau_1 > t\}$ is an atom of $\sigma(\tau_1)$, it follows that

$$\mathbf{1}_{\{\tau_1 > t, \tau_2 \le t\}} \frac{\varphi_t(h_1(t); Z_1)}{\varphi_t(h_1(t); 1)} = \mathbf{1}_{\{\tau_1 > t, \tau_2 \le t\}} \mathbb{E}[Z_1 \mid \xi^1_t, h_2(\tau_2), \tau_1 > t] = \mathbf{1}_{\{\tau_1 > t, \tau_2 \le t\}} \mathbb{E}[Z_1 \mid \mathcal{G}_t].$$

We remark that Proposition 3.1 implies

$$dW^{(1|\mathbb{G})}_t = d\xi^1_t - \sigma_1 \mathbf{1}_{\{\tau_1 > t, \tau_2 \le t\}} \mathbb{E}[Z_1 | \mathcal{G}_t] dt.$$

Consequently, we can conclude that the second term of (14) satisfies

$$
\begin{aligned}
(18)\qquad & d\left(P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}\right) \\
&= \mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}D^{(1)}_{tT}\left\{\left(r_t+\lambda^{(1|\mathbb{G})}_t\right)dt+\sigma_1\Sigma^{(1|\mathbb{G})}_{tT}dW^{(1|\mathbb{G})}_t\right\} \\
&\quad + P_{tT}\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}\left(-\mathbf{1}_{\{\tau_2<t\}}d\mathbf{1}_{\{\tau_1\le t\}}+\mathbf{1}_{\{\tau_1\ge t\}}d\mathbf{1}_{\{\tau_2\le t\}}\right) \\
&= \mathbf{1}_{\{\tau_1>t,\tau_2\le t\}}D^{(1)}_{t-,T}\left\{\left(r_t+\lambda^{(1|\mathbb{G})}_t\right)dt+\sigma_1\Sigma^{(1|\mathbb{G})}_{tT}dW^{(1|\mathbb{G})}_t-d\mathbf{1}_{\{\tau_1\le t\}}\right\} \\
&\quad + \mathbf{1}_{\{\tau_1\ge t\}}P_{tT}\frac{\widehat{\varphi}_t(h_1(T),h_2(t);1)}{\widehat{\varphi}_t(h_1(t),h_2(t);1)}d\mathbf{1}_{\{\tau_2\le t\}}.
\end{aligned}
$$

The last equality follows from the fact that for $t=\tau_2$, we have

$$
\left.\frac{F^{(1|2)}_{t,T}}{F^{(1|2)}_{t,t}}\right|_{t=\tau_2}=\left.\frac{\mathbb{E}\left[\mathbf{1}_{\{Z_1>h_1(T)\}}\mid \xi^1_t, Z_2=h_2(t)\right]}{\mathbb{E}\left[\mathbf{1}_{\{Z_1>h_1(t)\}}\mid \xi^1_t, Z_2=h_2(t)\right]}\right|_{t=\tau_2}=\left.\frac{\widehat{\varphi}_t(h_1(T),h_2(t);1)}{\widehat{\varphi}_t(h_1(t),h_2(t);1)}\right|_{t=\tau_2}.
$$

4.2. **The dynamics of $\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}D^{(1)}_{tT}$.** Now, we focus on the first term of (14), in which case both obligors are still active at time t. The idea of the proof is almost the same as that of the previous subsection. First, we can show

$$
\begin{aligned}
d\Big(P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}\frac{F^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,tt}}\Big) &= r_tP_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}\frac{F^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,tt}}dt+P_{tT}\frac{F^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,tt}}d\mathbf{1}_{\{\tau_1>t,\tau_2>t\}} \\
&\quad + P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}d\Big(\frac{F^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,tt}}\Big) \\
&= r_tD^{(1)}_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}dt+P_{tT}\frac{F^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,tt}}\left(-\mathbf{1}_{\{\tau_2\ge t\}}d\mathbf{1}_{\{\tau_1\le t\}}-\mathbf{1}_{\{\tau_1\ge t\}}d\mathbf{1}_{\{\tau_2\le t\}}\right) \\
&\quad + P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}d\Big(\frac{F^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,tt}}\Big)
\end{aligned}
$$

Moreover, the last term of the right-hand side can be represented as follows:

$$
\begin{aligned}
(19)\qquad & P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}d\Big(\frac{F^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,tt}}\Big) \\
&= \mathbf{1}_{\{\tau_1>t,\tau_2>t\}}D^{(1)}_{tT}\left[\frac{dF^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,Tt}}-\frac{dF^{(1|\emptyset)}_{t,tt}}{F^{(1|\emptyset)}_{t,tt}}+\left(\frac{dF^{(1|\emptyset)}_{t,tt}}{F^{(1|\emptyset)}_{t,tt}}\right)^2-\frac{dF^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,Tt}}\frac{dF^{(1|\emptyset)}_{t,tt}}{F^{(1|\emptyset)}_{t,tt}}\right].
\end{aligned}
$$

Next, we can calculate the stochastic differential of (12) with $u_1 = T$ and $u_2 = t$ as

$$dF_{t,Tt}^{(1|\emptyset)} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} d\Big(\mathbf{1}_{\{h_1^{-1}(z_1)>T\}}\mathbf{1}_{\{h_2^{-1}(z_2)>t\}}e^{\sigma_1 z_1\xi_t^1-\frac{1}{2}\sigma_1^2 z_1^2 t}\cdot e^{\sigma_2 z_2\xi_t^2-\frac{1}{2}\sigma_2^2 z_2^2 t}\Big)p_0(z_1,z_2)dz_1dz_2$$
$$= \big\{-\psi_{t,2}(h_1(T),h_2(t))dt + \sigma_1\varphi_t(h_1(T),h_2(t);Z_1)d\xi_t^1 + \sigma_2\varphi_t(h_1(T),h_2(t);Z_2)d\xi_t^2\big\}$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_0(z_1,z_2)e^{\sigma_1 z_1\xi_t^1+\sigma_2 z_2\xi_t^2-\frac{1}{2}(\sigma_1^2 z_1^2+\sigma_2^2 z_2^2)t}dz_1dz_2,$$

where we remember the prior joint distribution of (Z_1, Z_2) is given by

$$p_0(z_1,z_2) = \frac{1}{2\pi\sqrt{1-\rho^2}}\exp\Big(-\frac{1}{2(1-\rho^2)}(z_1^2-2\rho z_1z_2+z_2^2)\Big).$$

Similarly, $F_{t,Tt}^{(1|\emptyset)}$ can be given as

$$F_{t,Tt}^{(1|\emptyset)} = \varphi_t(h_1(T),h_2(t);1)\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_0(z_1,z_2)e^{\sigma_1 z_1\xi_t^1+\sigma_2 z_2\xi_t^2-\frac{1}{2}(\sigma_1^2 z_1^2+\sigma_2^2 z_2^2)t}dz_1dz_2,$$

therefore, we have

$$\frac{dF_{t,Tt}^{(1|\emptyset)}}{F_{t,Tt}^{(1|\emptyset)}} = -\frac{\psi_{t,2}(h_1(T),h_2(t))}{\varphi_t(h_1(T),h_2(t);1)}dt + \sigma_1\frac{\varphi_t(h_1(T),h_2(t);Z_1)}{\varphi_t(h_1(T),h_2(t);1)}d\xi_t^1 + \sigma_2\frac{\varphi_t(h_1(T),h_2(t);Z_2)}{\varphi_t(h_1(T),h_2(t);1)}d\xi_t^2. \tag{20}$$

Then, we deal with $\frac{dF_{t,tt}^{(1|\emptyset)}}{F_{t,tt}^{(1|\emptyset)}}$. In the last case, we can calculate the stochastic differential of (12) with $u_1 = u_2 = t$, such as

$$dF_{t,tt}^{(1|\emptyset)} = \Big\{-\psi_{t,1}(h_1(t),h_2(t))dt - \psi_{t,2}(h_1(t),h_2(t))dt$$
$$+\sigma_1\varphi_t(h_1(t),h_2(t);Z_1)d\xi_t^1 + \sigma_2\varphi_t(h_1(t),h_2(t);Z_2)d\xi_t^2\Big\}$$
$$\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_0(z_1,z_2)e^{\sigma_1 z_1\xi_t^1+\sigma_2 z_2\xi_t^2-\frac{1}{2}(\sigma_1^2 z_1^2+\sigma_2^2 z_2^2)t}dz_1dz_2.$$

Therefore, we divide it $F_{t,tt}^{(1|\emptyset)}$, given by

$$F_{t,tt}^{(1|\emptyset)} = \varphi_t(h_1(t),h_2(t);1)\times\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} p_0(z_1,z_2)e^{\sigma_1 z_1\xi_t^1+\sigma_2 z_2\xi_t^2-\frac{1}{2}(\sigma_1^2 z_1^2+\sigma_2^2 z_2^2)t}dz_1dz_2,$$

to achieve

$$\frac{dF_{t,tt}^{(1|\emptyset)}}{F_{t,tt}^{(1|\emptyset)}} = -\underbrace{\frac{\psi_{t,1}(h_1(t),h_2(t))}{\varphi_t(h_1(t),h_2(t);1)}}_{=\mathbf{1}_{\{\tau_2>t\}}\lambda_t^{(1|\mathbb{G})}}dt \;-\; \frac{\psi_{t,2}(h_1(t),h_2(t))}{\varphi_t(h_1(t),h_2(t);1)}dt \tag{21}$$
$$+\sigma_1\frac{\varphi_t(h_1(t),h_2(t);Z_1)}{\varphi_t(h_1(t),h_2(t);1)}d\xi_t^1 + \sigma_2\frac{\varphi_t(h_1(t),h_2(t);Z_2)}{\varphi_t(h_1(t),h_2(t);1)}d\xi_t^2.$$

The first and the second terms can be regarded as the conditional hazard rate of the obligors 1 and 2, respectively; however, we must note that the condition with respect to obligor 1 in the second term is slightly different from that in (20). Thus, by substituting (20) and (21) into (19), we obtain

$$
\begin{aligned}
(22)\qquad & d\Big(P_{tT}\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}\frac{F^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,tt}}\Big)\\
&= \mathbf{1}_{\{\tau_1>t,\tau_2>t\}}D^{(1)}_{tT}\Bigg\{\big(r_t+\lambda^{(1|\mathbb{G})}_t+\eta^{(2|\mathbb{G})}_{tT}\big)dt \;+\; \sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t \;+\; \sigma_2\Sigma^{(2|\mathbb{G})}_{1:tT}dW^{(2|\mathbb{G})}_t\Bigg\}\\
&\qquad - P_{tT}\frac{F^{(1|\emptyset)}_{t,Tt}}{F^{(1|\emptyset)}_{t,tt}}\big(\mathbf{1}_{\{\tau_2\geq t\}}d\mathbf{1}_{\{\tau_1\leq t\}}+\mathbf{1}_{\{\tau_1\geq t\}}d\mathbf{1}_{\{\tau_2\leq t\}}\big)\\
&= \mathbf{1}_{\{\tau_1>t,\tau_2>t\}}D^{(1)}_{t-,T}\Bigg\{\big(r_t+\lambda^{(1|\mathbb{G})}_t+\eta^{(2|\mathbb{G})}_{tT}\big)dt\\
&\qquad + \;\sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t \;+\; \sigma_2\Sigma^{(2|\mathbb{G})}_{1:tT}dW^{(2|\mathbb{G})}_t - d\mathbf{1}_{\{\tau_1\leq t\}}\Bigg\}\\
&\qquad - P_{tT}\frac{\varphi_t(h_1(T),h_2(t);1)}{\varphi_t(h_1(t),h_2(t);1)}\mathbf{1}_{\{\tau_1\geq t\}}d\mathbf{1}_{\{\tau_2\leq t\}}.
\end{aligned}
$$

4.3. **The dynamics of $D^{(1)}_{tT}$.** Finally, we are now in a position to achieve the SDE for $D^{(1)}_{tT}$. Combining (22) and (18), it immediately follows that

$$
\begin{aligned}
dD^{(1)}_{tT} &= d\Bigg(\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}D^{(1)}_{tT} \;+\; \mathbf{1}_{\{\tau_1>t,\tau_2\leq t\}}D^{(1)}_{tT}\Bigg)\\
&= D^{(1)}_{t-,T}\Bigg\{\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}\Big[\big(r_t+\lambda^{(1|\mathbb{G})}_t+\eta^{(2|\mathbb{G})}_{tT}\big)dt+\sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t+\sigma_2\Sigma^{(2|\mathbb{G})}_{1:tT}dW^{(2|\mathbb{G})}_t-d\mathbf{1}_{\{\tau_1\leq t\}}\Big]\\
&\qquad + \mathbf{1}_{\{\tau_1>t,\tau_2\leq t\}}\Big[\big(r_t+\lambda^{(1|\mathbb{G})}_t\big)dt+\sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t-d\mathbf{1}_{\{\tau_1\leq t\}}\Big]\Bigg\}\\
&\qquad - \mathbf{1}_{\{\tau_1\geq t\}}P_{tT}\left(\frac{\varphi_t(h_1(T),h_2(t);1)}{\varphi_t(h_1(t),h_2(t);1)}-\frac{\widehat{\varphi}_t(h_1(T),h_2(t);1)}{\widehat{\varphi}_t(h_1(t),h_2(t);1)}\right)d\mathbf{1}_{\{\tau_2\leq t\}}\\
&= D^{(1)}_{t-,T}\Bigg\{\big(r_t+\lambda^{(1|\mathbb{G})}_t+\eta^{(2|\mathbb{G})}_{tT}\big)dt+\sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t+\sigma_2\Sigma^{(2|\mathbb{G})}_{1:tT}dW^{(2|\mathbb{G})}_t-d\mathbf{1}_{\{\tau_1\leq t\}}\Bigg\}\\
&\qquad - \mathbf{1}_{\{\tau_1\geq t\}}P_{tT}\left(\frac{\varphi_t(h_1(T),h_2(t);1)}{\varphi_t(h_1(t),h_2(t);1)}-\frac{\widehat{\varphi}_t(h_1(T),h_2(t);1)}{\widehat{\varphi}_t(h_1(t),h_2(t);1)}\right)d\mathbf{1}_{\{\tau_2\leq t\}}.
\end{aligned}
$$

Because we have from (14)

$$
D^{(1)}_{\tau_2-,T} = \mathbf{1}_{\{\tau_1\geq\tau_2\}}P_{\tau_2T}\frac{\varphi_{\tau_2}(h_1(T),h_2(\tau_2);1)}{\varphi_{\tau_2}(h_1(\tau_2),h_2(\tau_2);1)},
$$

we can obtain

$$
\begin{aligned}
&\mathbf{1}_{\{\tau_1\geq t\}}P_{tT}\left(\frac{\varphi_t(h_1(T),h_2(t);1)}{\varphi_t(h_1(t),h_2(t);1)}-\frac{\widehat{\varphi}_t(h_1(T),h_2(t);1)}{\widehat{\varphi}_t(h_1(t),h_2(t);1)}\right)d\mathbf{1}_{\{\tau_2\leq t\}}\\
&\quad = D^{(1)}_{t-,T}\left(1-\frac{\varphi_t(h_1(t),h_2(t);1)}{\varphi_t(h_1(T),h_2(t);1)}\frac{\widehat{\varphi}_t(h_1(T),h_2(t);1)}{\widehat{\varphi}_t(h_1(t),h_2(t);1)}\right)d\mathbf{1}_{\{\tau_2\leq t\}} = D^{(1)}_{t-,T}\left(1-\Xi^{(2|\mathbb{G})}_{tT}\right)d\mathbf{1}_{\{\tau_2\leq t\}}.
\end{aligned}
$$

Substituting this jump term, we can conclude

$$dD^{(1)}_{tT} = D^{(1)}_{t-,T}\Bigg\{\left(r_t + \lambda^{(1|\mathbb{G})}_t + \eta^{(2|\mathbb{G})}_{1:tT}\right)dt + \sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t + \sigma_2\Sigma^{(2|\mathbb{G})}_{1:tT}dW^{(2|\mathbb{G})}_t - d\mathbf{1}_{\{\tau_1\le t\}} - \left(1-\Xi^{(2|\mathbb{G})}_{1:tT}\right)d\mathbf{1}_{\{\tau_2\le t\}}\Bigg\},$$

and furthermore, to represent the martingale terms explicitly in our stochastic differential equation,

$$dD^{(1)}_{tT} = D^{(1)}_{t-,T}\Bigg\{r_t dt + \sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t + \sigma_2\Sigma^{(2|\mathbb{G})}_{1:tT}dW^{(2|\mathbb{G})}_t - \left(d\mathbf{1}_{\{\tau_1\le t\}} - \mathbf{1}_{\{\tau_1>t\}}\lambda^{(1|\mathbb{G})}_t dt\right) - \left(1-\Xi^{(2|\mathbb{G})}_{1:tT}\right)\left(d\mathbf{1}_{\{\tau_2\le t\}} - \mathbf{1}_{\{\tau_2>t\}}\lambda^{(2|\mathbb{G})}_t dt\right)\Bigg\}. \tag{23}$$

The equation of $D^{(2)}_{tT}$ can be obtained by interchanging the roles of obligors 1 and 2 likewise, and the proof of the Theorem 3.5 is complete. Finally, we remark that the expression $\mathbf{1}_{\{\tau_i>t\}}\lambda^{(i|\mathbb{G})}_t$ is appropriate to represent clearly that the intensity $\lambda^{(i|\mathbb{G})}_t$ vanishes after τ_i, however, we sometimes omit the indicator process.

As for the first equation in Theorem 3.5, we should remark that on the set $\{\tau_2 > t\}$,

$$\begin{aligned}
\eta^{(2|\mathbb{G})}_{tT} &= \frac{\psi_{t,2}(h_1(t),h_2(t))}{\varphi_t(h_1(t),h_2(t);1)} - \frac{\psi_{t,2}(h_1(T),h_2(t))}{\varphi_t(h_1(T),h_2(t);1)} \\
&= \frac{\psi_{t,2}(h_1(t),h_2(t))}{\varphi_t(h_1(t),h_2(t);1)}\left(1 - \frac{\varphi_t(h_1(t),h_2(t);1)}{\psi_{t,2}(h_1(t),h_2(t))}\frac{\psi_{t,2}(h_1(T),h_2(t))}{\varphi_t(h_1(T),h_2(t);1)}\right) \\
&= \lambda^{(2|\mathbb{G})}_t\left(1 - \frac{\varphi_t(h_1(t),h_2(t);1)}{\varphi_t(h_1(T),h_2(t);1)}\frac{\psi_{t,2}(h_1(T),h_2(t))}{\psi_{t,2}(h_1(t),h_2(t))}\right) \\
&= \lambda^{(2|\mathbb{G})}_t\left(1 - \frac{\varphi_t(h_1(t),h_2(t);1)}{\varphi_t(h_1(T),h_2(t);1)}\frac{\widehat{\varphi}_{t,1}(h_1(T),h_2(t);1)}{\widehat{\varphi}_{t,1}(h_1(t),h_2(t);1)}\right) = \lambda^{(2|\mathbb{G})}_t\left(1-\Xi^{(2|\mathbb{G})}_{tT}\right).
\end{aligned}$$

The second last equality follows from

$$\begin{aligned}
\frac{\psi_{t,2}(h_1(T),h_2(t))}{\psi_{t,2}(h_1(t),h_2(t))} &= \frac{\mathbb{P}\left(Z_1 > h_1(T), Z_2 = h_2(t) \mid \xi^1_t,\xi^2_t\right)}{\mathbb{P}\left(Z_1 > h_1(t), Z_2 = h_2(t) \mid \xi^1_t,\xi^2_t\right)} \\
&= \frac{\mathbb{P}\left(Z_1 > h_1(T) \mid \xi^1_t,\xi^2_t, Z_2 = h_2(t)\right)\mathbb{P}\left(Z_2 = h_2(t) \mid \xi^1_t,\xi^2_t\right)}{\mathbb{P}\left(Z_1 > h_1(t) \mid \xi^1_t,\xi^2_t, Z_2 = h_2(t)\right)\mathbb{P}\left(Z_2 = h_2(t) \mid \xi^1_t,\xi^2_t\right)} \\
&= \frac{\mathbb{P}\left(Z_1 > h_1(T) \mid \xi^1_t,\xi^2_t, Z_2 = h_2(t)\right)}{\mathbb{P}\left(Z_1 > h_1(t) \mid \xi^1_t,\xi^2_t, Z_2 = h_2(t)\right)} = \frac{\mathbb{P}\left(Z_1 > h_1(T) \mid \xi^1_t, Z_2 = h_2(t)\right)}{\mathbb{P}\left(Z_1 > h_1(t) \mid \xi^1_t, Z_2 = h_2(t)\right)} \\
&= \frac{\mathbb{E}\left[\mathbf{1}_{\{Z_1>h_1(T)\}}\cdot 1 \mid \xi^1_t, Z_2 = h_2(t)\right]}{\mathbb{E}\left[\mathbf{1}_{\{Z_1>h_1(t)\}}\cdot 1 \mid \xi^1_t, Z_2 = h_2(t)\right]} = \frac{\widehat{\varphi}_{t,1}(h_1(T),h_2(t);1)}{\widehat{\varphi}_{t,1}(h_1(t),h_2(t);1)}.
\end{aligned}$$

Similarly, the SDE of the defalutable bond price process $\{D^{(2)}_{tT}\}$ issued by obligor 2 can be obtained by exchanging the roles between the two obligors. It is therefore of interest to examine the interaction between $D_{tT}^{(1)}$ and $D_{tT}^{(2)}$. To be more specific,

$$\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}\mathrm{Corr}\left(\frac{dD_{tT}^{(1)}}{D_{t-,T}^{(1)}}\ ,\ \frac{dD_{tT}^{(2)}}{D_{t-,T}^{(2)}}\right)=\mathbf{1}_{\{\tau_1>t,\tau_2>t\}}\left(\sigma_1^2\Sigma_{1:tT}^{(1|\emptyset)}\Sigma_{2:tT}^{(1|\emptyset)}+\sigma_2^2\Sigma_{1:tT}^{(2|\emptyset)}\Sigma_{2:tT}^{(2|\emptyset)}\right),$$

with simplified notations such that

$$\Sigma_{2:tT}^{(1|\emptyset)}:=\frac{\varphi_t(h_1(t),h_2(T);Z_1)}{\varphi_t(h_1(t),h_2(T);1)}-\frac{\varphi_t(h_1(t),h_2(t);Z_1)}{\varphi_t(h_1(t),h_2(t);1)},$$
$$\Sigma_{2:tT}^{(2|\emptyset)}:=\frac{\varphi_t(h_1(t),h_2(T);Z_2)}{\varphi_t(h_1(t),h_2(T);1)}-\frac{\varphi_t(h_1(t),h_2(t);Z_2)}{\varphi_t(h_1(t),h_2(t);1)}.$$

It is easy to see that $\Sigma_{k:tT}^{(i|\mathbb{G})}$ is not dependent on ρ by verifying $\frac{\partial}{\partial\rho}\Sigma_{k:tT}^{(i|\mathbb{G})}=0$. Besides the above covariation part, the trend term of $D_{tT}^{(1)}$ and $D_{tT}^{(2)}$ interact with each other through the additional term $\eta_{tT}^{(1|\mathbb{G})}$ and $\eta_{tT}^{(2|\mathbb{G})}$.

5. Numerical illustrations

Now we recall from Theorem 3.5 that the trend term of one defaultable discount bond process $\{D_{tT}^{(1)}\}$ contains not only its own hazard rate $\lambda_t^{(1|\mathbb{G})}$ but also the quantity $\eta_{tT}^{(2|\mathbb{G})}=\lambda_t^{(2|\mathbb{G})}(1-\Xi_{tT}^{(2|\mathbb{G})})$. Furthermore we show in Proposition 3.6 that the sign of $\eta_{tT}^{(2|\mathbb{G})}$ coincides with that of the correlation parameter ρ. Thus, one question now arises: how are the counterpart's contribution $\eta_{tT}^{(2|\mathbb{G})}$ to the trend term and the pseudo-recovery rate $\Xi_{tT}^{(2|\mathbb{G})}$ induced by the default of obligor 2 related to the correlation parameter ρ?

In this section, though limited to some parameter sets, we investigate this question numerically. Basically, we utilize the same structure of default times presented at the end of section 2, namely, we assume that the default time is specified by $\tau_i=h_i^{-1}(Z_i):=-\log\big(\Phi(-Z_i)\big)/\bar{\lambda}_i$ with $\bar{\lambda}_1=0.02,\bar{\lambda}_2=0.05$. For our numerical calculations, we fix $t=0.5$ and $T=1$ hereafter. Moreover, we assume no default case, in short, $\tau_1>t$ and $\tau_2>t$ at a given time t, and simply $\eta_{tT}^{(2|\emptyset)}$ (resp. $\lambda_t^{(2|\emptyset)}$) denotes $\eta_{tT}^{(2|\mathbb{G})}$ (resp. $\lambda_t^{(2|\mathbb{G})}$) for the no default case. Then we have:

$$\eta_{tT}^{(2|\emptyset)}=\underbrace{\frac{\mathbb{P}\big(Z_1>h_1(t),Z_2=h_2(t)\big|\xi_t^1,\xi_t^2\big)}{\mathbb{P}\big(Z_1>h_1(t),Z_2>h_2(t)\big|\xi_t^1,\xi_t^2\big)}}_{=\lambda_t^{(2|\emptyset)}}-\frac{\mathbb{P}\big(Z_1>h_1(T),Z_2=h_2(t)\big|\xi_t^1,\xi_t^2\big)}{\mathbb{P}\big(Z_1>h_1(T),Z_2>h_2(t)\big|\xi_t^1,\xi_t^2\big)}\tag{24}$$
$$=\frac{\int_{h_1(t)}^\infty e^{-\frac{1}{2(1-\rho^2)}(z_1^2-2\rho z_1h_2(t)+h_2^2(t))}e^{\sigma_1z_1\xi_t^1-\frac12\sigma_1^2z_1^2t}e^{\sigma_2h_2(t)\xi_t^2-\frac12\sigma_2^2h_2^2(t)t}dz_1}{\int_{h_1(t)}^\infty\int_{h_2(t)}^\infty e^{-\frac{1}{2(1-\rho^2)}(z_1^2-2\rho z_1z_2+z_2^2)}e^{\sigma_1z_1\xi_t^1-\frac12\sigma_1^2z_1^2t}e^{\sigma_2z_2\xi_t^2-\frac12\sigma_2^2z_2^2t}dz_1dz_2}$$
$$-\frac{\int_{h_1(T)}^\infty e^{-\frac{1}{2(1-\rho^2)}(z_1^2-2\rho z_1h_2(t)+h_2^2(t))}e^{\sigma_1z_1\xi_t^1-\frac12\sigma_1^2z_1^2t}e^{\sigma_2h_2(t)\xi_t^2-\frac12\sigma_2^2h_2^2(t)t}dz_1}{\int_{h_1(T)}^\infty\int_{h_2(t)}^\infty e^{-\frac{1}{2(1-\rho^2)}(z_1^2-2\rho z_1z_2+z_2^2)}e^{\sigma_1z_1\xi_t^1-\frac12\sigma_1^2z_1^2t}e^{\sigma_2z_2\xi_t^2-\frac12\sigma_2^2z_2^2t}dz_1dz_2}.$$

Hence, the formula can be seen as a deterministic function of $\rho, \sigma_1, \sigma_2, \xi_t^1(\omega), \xi_t^2(\omega)$ for preliminarily fixed $t = 0.5$, and $T = 1$. With these results in mind, we illustrate the relation between correlation ρ and obligor 2's conditional hazard rate for no default case $\eta_{tT}^{(2|\emptyset)}$ (and pseudo-recovery rate $\Xi_{tT}^{(2|\mathbb{G})} = 1 - \eta_{tT}^{(2|\emptyset)}/\lambda_t^{(2|\emptyset)}$) by numerically computing (24) for $\rho \in [-0.9, 0.9]$ and for each $\xi_t^1(\omega) = 0.1, 0.3$ and 0.5. Numerical integration is performed using MATLAB.

First, we assume $\sigma_1 = \sigma_2 = 1$, and $\xi_t^2(\omega) = 0$ as a most-likely scenario. Figure 2 presents the curves of $\eta_{tT}^{(2|\emptyset)}$ and $\Xi_{tT}^{(2|\mathbb{G})}$ under the assumptions. If $\rho < 0$, $\eta_{tT}^{(2|\emptyset)} < 0$, and vice versa, in a remarkably nonlinear way. We observe that for $\rho > 0$, the larger ρ is, the larger $\eta_{tT}^{(2|\emptyset)}$ is. In contrast, in the case

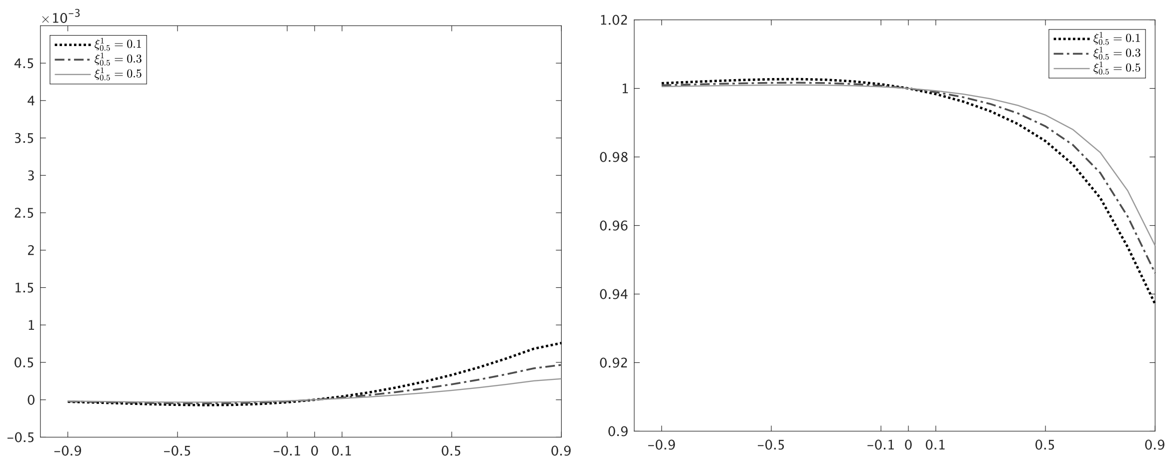

FIGURE 2. Under the assumption of $\sigma_1 = \sigma_2 = 1$, and $\xi_t^2(\omega) = 0$ as a most-likely scenario, the curves of obligor 2's hazard rate difference $\eta_{tT}^{(2|\emptyset)}$(left panel) and the pseudo-recovery rate $\Xi_{tT}^{(2|\emptyset)}$ (right panel) with the correlation parameter $\rho = Corr(Z_1, Z_2)$ for $\xi_t^1 = 0.1, 0.3$ and 0.5.

of $\rho < 0$, it seems that there exists a lower bound. In addition, we can see that because the value ξ_t^1 of the information flow contributes to the bond price $D_{tT}^{(1)}$ positively, $\eta_{tT}^{(2|\emptyset)}$ decreases its absolute value as $\xi_t^1(\omega)$ increases. In contrast, the graph of $\Xi_{tT}^{(2|\mathbb{G})}$ shows the fractional recovery of the market value at which the bond price $D_{\tau_2-,T}^{(1)}$ jumps to $D_{\tau_2,T}^{(1)} = \Xi_{\tau_2-,T}^{(1|\emptyset)} D_{\tau_2-,T}^{(1)}$ due to the default of obligor 2. In the case of $\rho > 0$, the larger ρ is, the larger the negative impact of the default is. However, a negative correlation makes a relatively small positive impact to its market value.

Second, we consider the case where informational uncertainty is more than the previous case, that is, the information flow rates σ_1 and σ_2 are less than those in the previous case. Figure 3 illustrates the results for the case of $\sigma_1 = \sigma_2 = 0.5$, and $\xi_t^2(\omega) = 0$. We notice that the absolute value of $\eta_{tT}^{(2|\emptyset)}$ becomes larger than that of the previous case. In addition, the nonlinearity with respect to ρ remains. In particular, for $\rho > 0$, one can see the fractional recovery of $D_{\tau_2,T}^{(1)}$ at default is lower than in the previous case.

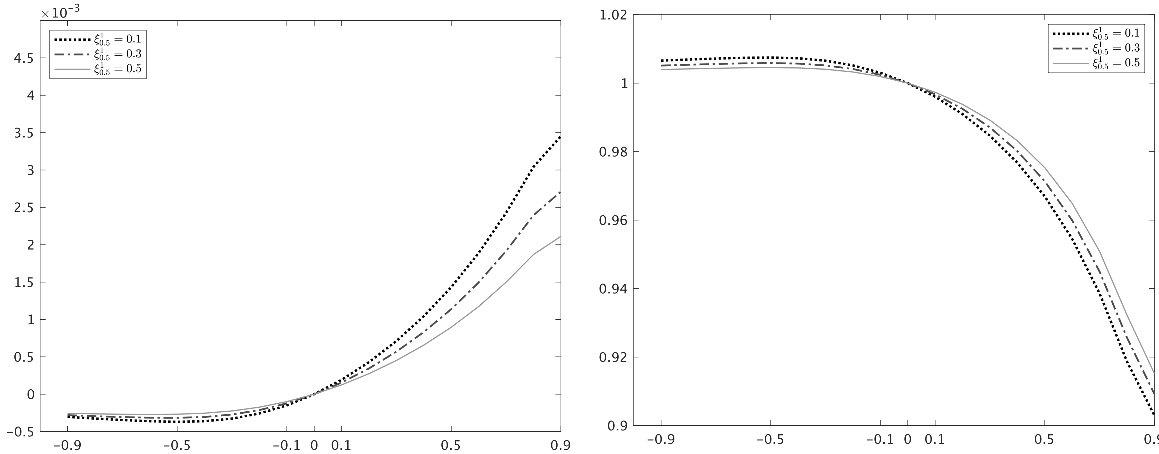

FIGURE 3. Under the assumption of $\sigma_1 = \sigma_2 = 0.5$ and $\xi^2_t(\omega) = 0$, the curves of obligor 2's hazard rate difference $\eta^{(2|\emptyset)}_{tT}$(left panel) and the pseudo-recovery rate $\Xi^{(2|\emptyset)}_{tT}$ (right panel) with the correlation parameter $\rho = Corr(Z_1, Z_2)$ for $\xi^1_t = 0.1, 0.3$ and 0.5.

Finally, we consider the case where the value of obligor 2's market information process is negative while the information flow rates σ_1 and σ_2 are the same as in the second case. Figure 4 illustrates the results for the case of $\sigma_1 = \sigma_2 = 0.5$ and $\xi^2_t(\omega) = -0.5$, which increases the absolute value of $\eta^{(2|\emptyset)}_{tT}$ in comparison with the second case. We remark that $\xi^2_t(\omega)$ contributes positively to the bond price $D^{(1)}_{tT}$; therefore, the negative value of $\xi^2_t(\omega)$ leads to a lower bond price, hence, a wider trend term than the second case. However, the shape of the curve $\Xi^{(2|\emptyset)}_{tT}$ is rarely different from the second case.

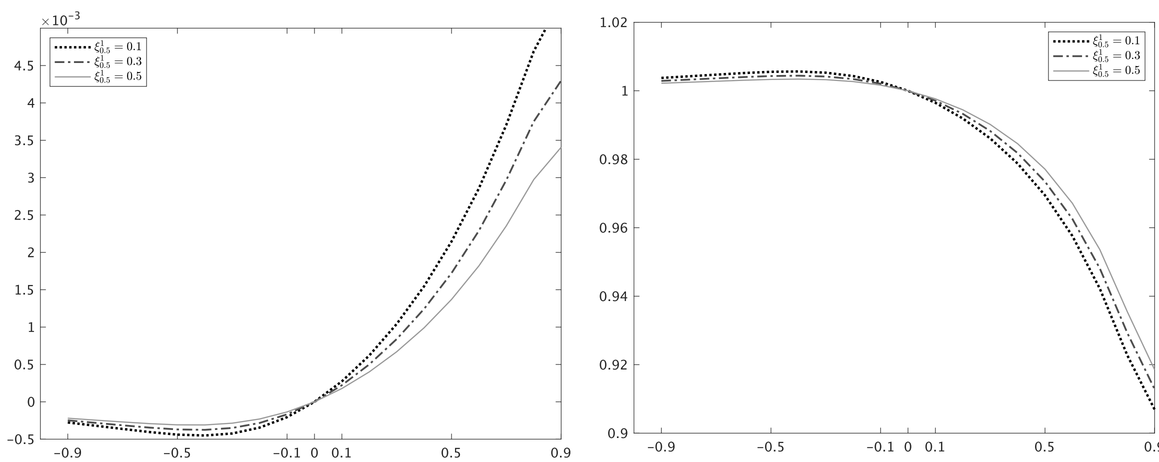

FIGURE 4. Under the assumption of $\sigma_1 = \sigma_2 = 0.5$, and $\xi^2_t(\omega) = -0.5$, the curves of obligor 2's hazard rate difference $\eta^{(2|\emptyset)}_{tT}$(left panel) and the pseudo-recovery rate $\Xi^{(2|\emptyset)}_{tT}$ (right panel) with the correlation parameter $\rho = Corr(Z_1, Z_2)$ for $\xi^1_t = 0.1, 0.3$ and 0.5.

Furthermore, we display some sample paths of the hazard rate processes to illustrate sudden jumps caused by the transfer from $\lambda^{(1|\emptyset)}_{\tau_2-} + \eta^{(2|\emptyset)}_{\tau_2-,T}$ to $\lambda^{(1|2)}_{\tau_2}$ at time τ_2. Obligor 1's hazard rate after the default of the counterpart is specified by

$$\lambda^{(1|2)}_t = \frac{\mathbb{P}\big(Z_1 = h_1(t)|\xi^1_t, \tau_2\big)}{\mathbb{P}\big(Z_1 > h_1(t)|\xi^1_t, \tau_2\big)} = \frac{e^{-\frac{1}{2(1-\rho^2)}(h_1(t)-\rho h_2(\tau_2))^2} e^{\sigma_1 h_1(t)\xi^1_t - \frac{1}{2}\sigma_1^2 h_1^2(t)t}}{\int_{h_1(t)}^{\infty} e^{-\frac{1}{2(1-\rho^2)}(z_1-\rho h_2(\tau_2))^2} e^{\sigma_1 z_1 \xi^1_t - \frac{1}{2}\sigma_1^2 z_1^2 t} dz_1}. \tag{25}$$

To numerically observe the impact of switching the default hazard rate before and after the default of counterpart, we simulate the trajectory of $\lambda^{(1|\emptyset)}_t + \eta^{(2|\emptyset)}_{tT}$ (before τ_2) and $\lambda^{(1|2)}_t$ (after τ_2) with the parameter set used in Figure 1 of Section 2.2 and the same assumption that obligor 2 defaults first at fixed time $\tau_2 = 0.5$. The calculations are based on (24), (25) and

$$\begin{aligned}
\lambda^{(1|\emptyset)}_t &= \frac{\mathbb{P}\big(Z_1 = h_1(t), Z_2 > h_2(t)|\xi^1_t, \xi^2_t\big)}{\mathbb{P}\big(Z_1 > h_1(t), Z_2 > h_2(t)|\xi^1_t, \xi^2_t\big)} \\
&= \frac{\int_{h_2(t)}^{\infty} e^{-\frac{1}{2(1-\rho^2)}(h_1^2(t)-2\rho h_1(t) z_2 + z_2^2)} e^{\sigma_1 h_1(t)\xi^1_t - \frac{1}{2}\sigma_1^2 h_1^2(t)t} e^{\sigma_2 z_2 \xi^2_t - \frac{1}{2}\sigma_2^2 z_2^2 t} dz_2}{\int_{h_1(t)}^{\infty}\int_{h_2(t)}^{\infty} e^{-\frac{1}{2(1-\rho^2)}(z_1^2-2\rho z_1 z_2 + z_2^2)} e^{\sigma_1 z_1 \xi^1_t - \frac{1}{2}\sigma_1^2 z_1^2 t} e^{\sigma_2 z_2 \xi^2_t - \frac{1}{2}\sigma_2^2 z_2^2 t} dz_1 dz_2}.
\end{aligned}$$

In Figure 5, we illustrate some simulated sample trajectories of $\{\lambda^{(1|\emptyset)}_t + \eta^{(2|\emptyset)}_{tT}\}_{0\le t<\tau_2}$ and $\{\lambda^{(1|2)}_t\}_{\tau_2\le t\le 1}$ for the relatively high correlation case of $\rho = 0.8$ (left panel) and the moderate correlation case of $\rho = 0.4$ (right panel). Similar to Figure 1, we see that the size of the upward jump of the hazard process is larger for the highly correlated case than for the moderately correlated case.

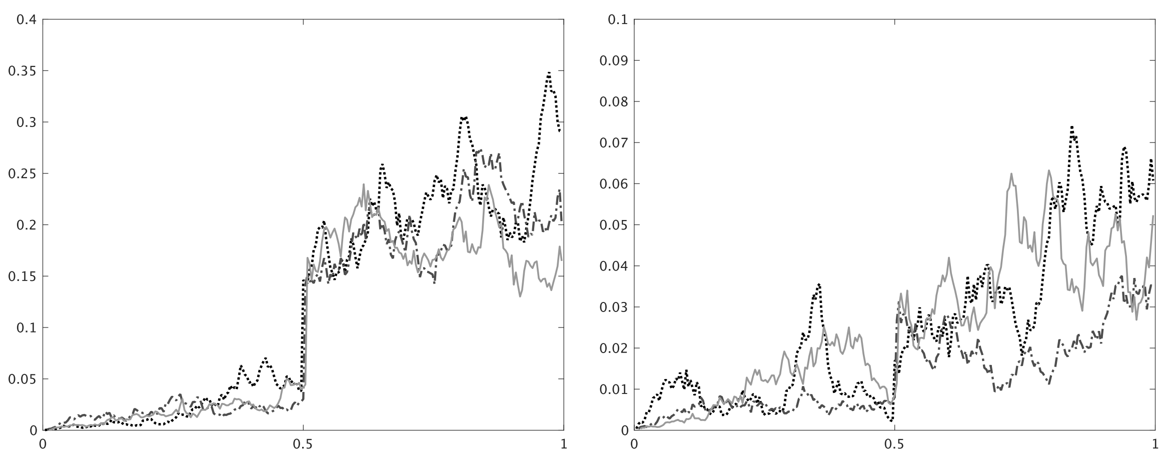

FIGURE 5. Simulated sample trajectories on the interval $[0, 1]$ of obligor 1's hazard rate process that switches from $\lambda^{(1|\emptyset)}_t + \eta^{(2|\emptyset)}_{tT}$ to $\lambda^{(1|2)}_t$ at fixed default time $\tau_2 = 0.5$ of obligor 2. The case of $\rho = 0.8$ (left panel with vertical axis $[0, 0.4]$) and the case of $\rho = 0.4$ (right panel with vertical axis $[0, 0.1]$).

6. Conclusion

We construct the default contagion model for a more advanced pricing of defaultable financial securities by extending the market information flow-based model proposed by Brody et al. (2010) to a multi-name case. In our default contagion model, the default time of each obligor is specified by the market factor associated with the obligor. The market factors are supposed to follow a multidimensional correlated normal distribution. However, market factors cannot be observed through the market unless the associated default happens; instead, we can utilize the obligors' market information processes specified by the market factors with independent Brownian noises until the associated default happens.

To evaluate the defaultable discount bonds under the model, we first obtain the conditional probabilities of default times given the available information generated by the history of market information processes of surviving obligors and the identified market factor of defaulted ones. In particular, we obtain some explicit representations for the case of two correlated obligors. Then, as a main result, we aim to derive the stochastic differential equation followed by one defaultable discount bond price process.

We explicitly show the derived equation only for the case of two correlated obligors in the theorem to avoid too complicated a representation. (Appendix mentions the case of three correlated obligors.) At first glance, the dynamics and the components seem to be complicated, but we see that the dynamics can be regarded as natural extensions of the previous models.

In one representation of the bond price dynamics, we notice that the time trend term of the bond price, before the counterpart obligor's default, includes the counterpart obligor's hazard rate adjusted with the "pseudo-default loss" rate as well as the issuer's hazard rate. In addition, the bond price can jump at the counterpart obligor's default time since the available information for pricing is largely updated by revealing the latent market factor of the counterpart, although the bond does not default due to the counterpart obligor's default.

The other representation is consistent with the martingale-based methods for credit risk modeling. Specifically, such a representation reveals that the defaultable bond price process is driven continuously by Brownian motions derived from both obligors' market information processes and can be jumped due to the martingales, defined as the default indicator processes compensated with the default intensity process.

If it happens before the maturity or the issuer's default, the stochastic drivers of defaultable bond price dynamics and the components, such as the issuer's hazard rate and volatility, are different before and after the counterpart obligor's default. Since the market factor of the counterpart is cleared at the very moment of the counterpart's default, what generates the available information is transferred and improved from the market information flows of both issuers to that of the surviving issuer and the true value of the market factor for the defaulted counterpart.

Finally, we conduct calculations because it is useful to visually determine the quantitative effects of counterpart obligors' default on the model components of the issuer. Indeed, we present some numerical illustrations for visualizing the theoretical consequence on the relation between the conditional default intensities and the market factor correlation parameter as well as the upward impact of counterpart obligors' default on the issuer's hazard rate. Concurrently, we can show that our model is tractable for numerical works. However, there are still issues for the practical use of our model, and we will make them assignments for future research.

Acknowledgments

This study was supported by JSPS KAKENHI Grant Number JP20K04960. We are grateful to the participants of the JSIAM 2020 Conference (Online) and Osaka Webinar on Mathematical Finance (Oct. 2020) for valuable discussions, especially for Jun Sekine. Moreover, the author expresses his gratitude toward anonymous referees for their helpful suggestions. We would also like to thank Editage (www.editage.com) for English language editing.

Appendix

In this appendix, we summarize some results for the $n = 3$ case of Theorem 3.5. Using the generalized Dellacherie formula shown in Proposition 2.5, the defaultable discount bond price of obligor 1 is given by

$$\begin{aligned} D^{(1)}_{t,T} = P_{t,T} \Bigg\{ & \mathbf{1}_{\{\tau_1>t,\tau_2>t,\tau_3>t\}} \frac{\mathbb{P}(\tau_1 > T, \tau_2 > t, \tau_3 > t \mid \xi^1_t, \xi^2_t, \xi^3_t)}{\mathbb{P}(\tau_1 > t, \tau_2 > t, \tau_3 > t \mid \xi^1_t, \xi^2_t, \xi^3_t)} \\ & + \mathbf{1}_{\{\tau_1>t,\tau_2\le t,\tau_3>t\}} \frac{\mathbb{P}(\tau_1 > T, \tau_2 \le t, \tau_3 > t \mid \xi^1_t, Z_2, \xi^3_t)}{\mathbb{P}(\tau_1 > t, \tau_2 \le t, \tau_3 > t \mid \xi^1_t, Z_2, \xi^3_t)} \\ & + \mathbf{1}_{\{\tau_1>t,\tau_2>t,\tau_3\le t\}} \frac{\mathbb{P}(\tau_1 > T, \tau_2 > t, \tau_3 \le t \mid \xi^1_t, \xi^2_t, Z_3)}{\mathbb{P}(\tau_1 > t, \tau_2 > t, \tau_3 \le t \mid \xi^1_t, \xi^2_t, Z_3)} \\ & + \mathbf{1}_{\{\tau_1>t,\tau_2\le t,\tau_3\le t\}} \frac{\mathbb{P}(\tau_1 > T, \tau_2 \le t, \tau_3 \le t \mid \xi^1_t, Z_2, Z_3)}{\mathbb{P}(\tau_1 > t, \tau_2 \le t, \tau_3 \le t \mid \xi^1_t, Z_2, Z_3)} \Bigg\}. \end{aligned}$$

From some calculations similar to those in Section 4, one sees that

$$
\begin{aligned}
\mathbf{1}_{\{\tau_1>t\}}dD^{(1)}_{tT} &= d\Big(\mathbf{1}_{\{\tau_1>t,\tau_2>t,\tau_3>t\}}D^{(1)}_{tT} + \mathbf{1}_{\{\tau_1>t,\tau_2\le t,\tau_3>t\}}D^{(1)}_{tT} + \mathbf{1}_{\{\tau_1>t,\tau_2>t,\tau_3\le t\}}D^{(1)}_{tT} + \mathbf{1}_{\{\tau_1>t,\tau_2\le t,\tau_3\le t\}}D^{(1)}_{tT}\Big) \\
&= D^{(1)}_{t-,T}\Big\{\mathbf{1}_{\{\tau_1>t,\tau_2>t,\tau_3>t\}}\Big[\Big(r_t+\lambda^{(1|\mathbb{G})}_t+\eta^{(2|\mathbb{G})}_{1:tT}+\eta^{(3|\mathbb{G})}_{1:tT}\Big)dt \\
&\quad + \sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t + \sigma_2\Sigma^{(2|\mathbb{G})}_{1:tT}dW^{(2|\mathbb{G})}_t + \sigma_3\Sigma^{(3|\mathbb{G})}_{1:tT}dW^{(3|\mathbb{G})}_t - d\mathbf{1}_{\{\tau_1\le t\}}\Big] \\
&\quad + \mathbf{1}_{\{\tau_1>t,\tau_2\le t,\tau_3>t\}}\Big[\big(r_t+\lambda^{(1|\mathbb{G})}_t+\eta^{(3|\mathbb{G})}_{1:tT}\big)dt + \sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t + \sigma_3\Sigma^{(3|\mathbb{G})}_{1:tT}dW^{(3|\mathbb{G})}_t - d\mathbf{1}_{\{\tau_1\le t\}}\Big] \\
&\quad + \mathbf{1}_{\{\tau_1>t,\tau_2>t,\tau_3\le t\}}\Big[\big(r_t+\lambda^{(1|\mathbb{G})}_t+\eta^{(2|\mathbb{G})}_{1:tT}\big)dt + \sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t + \sigma_2\Sigma^{(2|\mathbb{G})}_{1:tT}dW^{(2|\mathbb{G})}_t - d\mathbf{1}_{\{\tau_1\le t\}}\Big] \\
&\quad + \mathbf{1}_{\{\tau_1>t,\tau_2\le t,\tau_3\le t\}}\Big[\big(r_t+\lambda^{(1|\mathbb{G})}_t\big)dt + \sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t - d\mathbf{1}_{\{\tau_1\le t\}}\Big]\Big\} \\
&\quad - \mathbf{1}_{\{\tau_1\ge t,\tau_3\ge t\}}P_{tT}\left(\frac{\varphi_{t,\{1,2,3\}}(h_1(T),h_2(t),h_3(t);1)}{\varphi_{t,\{1,2,3\}}(h_1(t),h_2(t),h_3(t);1)} - \frac{\varphi_{t,\{1,3\}}(h_1(T),h_2(t),h_3(t);1)}{\varphi_{t,\{1,3\}}(h_1(t),h_2(t),h_3(t);1)}\right)d\mathbf{1}_{\{\tau_2\le t\}} \\
&\quad - \mathbf{1}_{\{\tau_1\ge t,\tau_2\ge t\}}P_{tT}\left(\frac{\varphi_{t,\{1,2,3\}}(h_1(T),h_2(t),h_3(t);1)}{\varphi_{t,\{1,2,3\}}(h_1(t),h_2(t),h_3(t);1)} - \frac{\varphi_{t,\{1,2\}}(h_1(T),h_2(t),h_3(t);1)}{\varphi_{t,\{1,2\}}(h_1(t),h_2(t),h_3(t);1)}\right)d\mathbf{1}_{\{\tau_3\le t\}} \\
&\quad - \mathbf{1}_{\{\tau_1\ge t,\tau_3\le t\}}P_{tT}\left(\frac{\varphi_{t,\{1,2\}}(h_1(T),h_2(t),h_3(\tau_3);1)}{\varphi_{t,\{1,2\}}(h_1(t),h_2(t),h_3(\tau_3);1)} - \frac{\varphi_{t,\{1\}}(h_1(T),h_2(t),h_3(\tau_3);1)}{\varphi_{t,\{1\}}(h_1(t),h_2(t),h_3(\tau_3);1)}\right)d\mathbf{1}_{\{\tau_2\le t\}} \\
&\quad - \mathbf{1}_{\{\tau_1\ge t,\tau_2\le t\}}P_{tT}\left(\frac{\varphi_{t,\{1,3\}}(h_1(T),h_2(\tau_2),h_3(t);1)}{\varphi_{t,\{1,3\}}(h_1(t),h_2(\tau_2),h_3(t);1)} - \frac{\varphi_{t,\{1\}}(h_1(T),h_2(\tau_2),h_3(t);1)}{\varphi_{t,\{1\}}(h_1(t),h_2(\tau_2),h_3(t);1)}\right)d\mathbf{1}_{\{\tau_3\le t\}},
\end{aligned}
$$

where

$$
\begin{aligned}
\lambda^{(1|\mathbb{G})}_t &:= \mathbf{1}_{\{\tau_2>t,\tau_3>t\}}\frac{\psi_{t,1,\{1,2,3\}}(h_1(t),h_2(t),h_3(t))}{\varphi_{t,\{1,2,3\}}(h_1(t),h_2(t),h_3(t);1)} + \mathbf{1}_{\{\tau_2\le t,\tau_3>t\}}\frac{\psi_{t,1,\{1,3\}}(h_1(t),h_2(\tau_2),h_3(t))}{\varphi_{t,\{1,3\}}(h_1(t),h_2(\tau_2),h_3(t);1)} \\
&\quad + \mathbf{1}_{\{\tau_2>t,\tau_3\le t\}}\frac{\psi_{t,1,\{1,2\}}(h_1(t),h_2(t),h_3(\tau_3))}{\varphi_{t,\{1,2\}}(h_1(t),h_2(t),h_3(\tau_3);1)} + \mathbf{1}_{\{\tau_2\le t,\tau_3\le t\}}\frac{\psi_{t,1,\{1\}}(h_1(t),h_2(\tau_2),h_3(\tau_3))}{\varphi_{t,\{1\}}(h_1(t),h_2(\tau_2),h_3(\tau_3);1)},
\end{aligned}
$$

is the hazard rate of the issuer (obligor 1), which is dependent on the global filtration $\mathbb{G} = (\mathcal{G}_t)_{t\ge 0}$,

$$
\begin{aligned}
\eta^{(i|\mathbb{G})}_{1:tT} &:= \mathbf{1}_{\{\tau_2>t,\tau_3>t\}}\Big(\frac{\psi_{t,i,\{1,2,3\}}(h_1(t),h_2(t),h_3(t))}{\varphi_{t,\{1,2,3\}}(h_1(t),h_2(t),h_3(t);1)} - \frac{\psi_{t,i,\{1,2,3\}}(h_1(T),h_2(t),h_3(t))}{\varphi_{t,\{1,2,3\}}(h_1(T),h_2(t),h_3(t);1)}\Big) \\
&\quad + \mathbf{1}_{\{\tau_2\le t,\tau_3>t\}}\Big(\frac{\psi_{t,i,\{1,3\}}(h_1(t),h_2(\tau_2),h_3(t))}{\varphi_{t,\{1,3\}}(h_1(t),h_2(\tau_2),h_3(t);1)} - \frac{\psi_{t,i,\{1,3\}}(h_1(T),h_2(\tau_2),h_3(t))}{\varphi_{t,\{1,3\}}(h_1(T),h_2(\tau_2),h_3(t);1)}\Big)\cdot\mathbf{1}_{\{i\ne 2\}} \\
&\quad + \mathbf{1}_{\{\tau_2>t,\tau_3\le t\}}\Big(\frac{\psi_{t,i,\{1,2\}}(h_1(t),h_2(t),h_3(\tau_3))}{\varphi_{t,\{1,2\}}(h_1(t),h_2(t),h_3(\tau_3);1)} - \frac{\psi_{t,i,\{1,2\}}(h_1(T),h_2(t),h_3(\tau_3))}{\varphi_{t,\{1,2\}}(h_1(T),h_2(t),h_3(\tau_3);1)}\Big)\cdot\mathbf{1}_{\{i\ne 3\}},
\end{aligned}
$$

is the hazard rate adjusted with the pseudo-default loss for obligor i ($i = 2$ or 3), and the fucntions $\varphi_{t,\mathcal{J}}$ and $\psi_{t,i,\mathcal{J}}$ are given in (10) and (11), respectively.

Moreover, the volatility components $\Sigma^{(i|\mathbb{G})}_{1:tT}$ $(i=1,2,3)$, which are also dependent on global filtration, are defined as

$$\begin{aligned}\Sigma^{(i|\mathbb{G})}_{1:tT} := \mathbf{1}_{\{\tau_2>t,\tau_3>t\}}&\left(\frac{\varphi_{t,\{1,2,3\}}(h_1(T),h_2(t),h_3(t);Z_i)}{\varphi_{t,\{1,2,3\}}(h_1(T),h_2(t),h_3(t);1)}-\frac{\varphi_{t,\{1,2,3\}}(h_1(t),h_2(t),h_3(t);Z_i)}{\varphi_{t,\{1,2,3\}}(h_1(t),h_2(t),h_3(t);1)}\right)\\
&+\mathbf{1}_{\{\tau_2\le t,\tau_3>t\}}\left(\frac{\varphi_{t,\{1,3\}}(h_1(T),h_2(\tau_2),h_3(t);Z_i)}{\varphi_{t,\{1,3\}}(h_1(T),h_2(\tau_2),h_3(t);1)}-\frac{\varphi_{t,\{1,3\}}(h_1(t),h_2(\tau_2),h_3(t);Z_i)}{\varphi_{t,\{1,3\}}(h_1(t),h_3(\tau_2),h_3(t);1)}\right)\cdot\mathbf{1}_{\{i\neq 2\}}\\
&+\mathbf{1}_{\{\tau_2>t,\tau_3\le t\}}\left(\frac{\varphi_{t,\{1,2\}}(h_1(T),h_2(t),h_3(\tau_3);Z_i)}{\varphi_{t,\{1,2\}}(h_1(T),h_2(t),h_3(\tau_3);1)}-\frac{\varphi_{t,\{1,2\}}(h_1(t),h_2(t),h_3(\tau_3);Z_i)}{\varphi_{t,\{1,2\}}(h_1(t),h_2(t),h_3(\tau_3);1)}\right)\cdot\mathbf{1}_{\{i\neq 3\}}\\
&+\mathbf{1}_{\{\tau_2\le t,\tau_3\le t\}}\left(\frac{\varphi_{t,\{1\}}(h_1(T),h_2(\tau_2),h_3(\tau_3);Z_1)}{\varphi_{t,\{1\}}(h_1(T),h_2(\tau_2),h_3(\tau_3);1)}-\frac{\varphi_{t,\{1\}}(h_1(t),h_2(\tau_2),h_3(\tau_3);Z_1)}{\varphi_{t,\{1\}}(h_1(t),h_2(\tau_2),h_3(\tau_3);1)}\right)\cdot\mathbf{1}_{\{i=1\}}.\end{aligned}$$

Thus, it follows from the second and the third line of the above that it does not necessarily satisfy $\mathbf{1}_{\{\tau_2\le t<\tau_3\}}\Sigma^{(1|\mathbb{G})}_{1:tT}=\mathbf{1}_{\{\tau_3\le t<\tau_2\}}\Sigma^{(1|\mathbb{G})}_{1:tT}$ because the order of defaults is different, while the equality $\mathbf{1}_{\{\tau_2<\tau_3\le t<\tau_1\}}\Sigma^{(1|\mathbb{G})}_{1:tT}=\mathbf{1}_{\{\tau_3<\tau_2\le t<\tau_1\}}\Sigma^{(1|\mathbb{G})}_{1:tT}$ holds true.

Finally we can wrap up the continuous part and then rewrite the jump part as follows:

$$\begin{aligned}\mathbf{1}_{\{\tau_1>t\}}dD^{(1)}_{tT}=D^{(1)}_{t-,T}\Bigg\{&\left(r_t+\lambda^{(1|\mathbb{G})}_t+\mathbf{1}_{\{\tau_2>t\}}\eta^{(2|\mathbb{G})}_{tT}+\mathbf{1}_{\{\tau_3>t\}}\eta^{(3|\mathbb{G})}_{tT}\right)dt\\
&+\sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t+\mathbf{1}_{\{\tau_2>t\}}\sigma_2\Sigma^{(2|\mathbb{G})}_{1:tT}dW^{(2|\mathbb{G})}_t+\mathbf{1}_{\{\tau_3>t\}}\sigma_3\Sigma^{(3|\mathbb{G})}_{1:tT}dW^{(3|\mathbb{G})}_t-d\mathbf{1}_{\{\tau_1\le t\}}\\
&-\Bigg(1-\underbrace{\frac{\varphi_{t,\{1,2,3\}}(h_1(t),h_2(t),h_3(t);1)}{\varphi_{t,\{1,2,3\}}(h_1(T),h_2(t),h_3(t);1)}\frac{\varphi_{t,\{1,3\}}(h_1(T),h_2(t),h_3(t);1)}{\varphi_{t,\{1,3\}}(h_1(t),h_2(t),h_3(t);1)}}_{\Xi^{(2|\emptyset)}_{1:tT}}\Bigg)d\mathbf{1}_{\{\tau_2\le t\}}\\
&-\Bigg(1-\underbrace{\frac{\varphi_{t,\{1,2,3\}}(h_1(t),h_2(t),h_3(t);1)}{\varphi_{t,\{1,2,3\}}(h_1(T),h_2(t),h_3(t);1)}\frac{\varphi_{t,\{1,2\}}(h_1(T),h_2(t),h_3(t);1)}{\varphi_{t,\{1,2\}}(h_1(t),h_2(t),h_3(t);1)}}_{\Xi^{(3|\emptyset)}_{1:tT}}\Bigg)d\mathbf{1}_{\{\tau_3\le t\}}\\
&-\Bigg(1-\underbrace{\frac{\varphi_{t,\{1,2\}}(h_1(t),h_2(t),h_3(\tau_3);1)}{\varphi_{t,\{1,2\}}(h_1(T),h_2(t),h_3(\tau_3);1)}\frac{\varphi_{t,\{1\}}(h_1(T),h_2(t),h_3(\tau_3);1)}{\varphi_{t,\{1\}}(h_1(t),h_2(t),h_3(\tau_3);1)}}_{\Xi^{(2|3)}_{1:tT}}\Bigg)d\mathbf{1}_{\{\tau_2\le t\}}\\
&-\Bigg(1-\underbrace{\frac{\varphi_{t,\{1,3\}}(h_1(t),h_2(\tau_2),h_3(t);1)}{\varphi_{t,\{1,3\}}(h_1(T),h_2(\tau_2),h_3(t);1)}\frac{\varphi_{t,\{1\}}(h_1(T),h_2(\tau_2),h_3(t);1)}{\varphi_{t,\{1\}}(h_1(t),h_2(\tau_2),h_3(t);1)}}_{\Xi^{(3|2)}_{1:tT}}\Bigg)d\mathbf{1}_{\{\tau_3\le t\}}\Bigg\}.\end{aligned}$$

Here, we set the pseudo-recovery rate of the pre-default market value as $\Xi^{(2|\emptyset)}_{1:tT}$, $\Xi^{(3|\emptyset)}_{1:tT}$, $\Xi^{(2|3)}_{1:tT}$ and $\Xi^{(3|2)}_{1:tT}$ depending on the default history. These are redefined with consistent notation $\Xi^{(i|\mathbb{G})}_{1:tT}$ as follows:

$$\begin{aligned}\Xi^{(2|\mathbb{G})}_{1:tT} := \mathbf{1}_{\{\tau_2>t,\tau_3>t\}} &\frac{\varphi_{t,\{1,2,3\}}(h_1(t),h_2(t),h_3(t);1)}{\varphi_{t,\{1,2,3\}}(h_1(T),h_2(t),h_3(t);1)}\frac{\varphi_{t,\{1,3\}}(h_1(T),h_2(t),h_3(t);1)}{\varphi_{t,\{1,3\}}(h_1(t),h_2(t),h_3(t);1)}\\ &+\mathbf{1}_{\{\tau_2>t,\tau_3\le t\}}\frac{\varphi_{t,\{1,2\}}(h_1(t),h_2(t),h_3(\tau_3);1)}{\varphi_{t,\{1,2\}}(h_1(T),h_2(t),h_3(\tau_3);1)}\frac{\varphi_{t,\{1\}}(h_1(T),h_2(t),h_3(\tau_3);1)}{\varphi_{t,\{1\}}(h_1(t),h_2(t),h_3(\tau_3);1)},\end{aligned}$$

$$\begin{aligned}\Xi^{(3|\mathbb{G})}_{1:tT} := \mathbf{1}_{\{\tau_2>t,\tau_3>t\}} &\frac{\varphi_{t,\{1,2,3\}}(h_1(t),h_2(t),h_3(t);1)}{\varphi_{t,\{1,2,3\}}(h_1(T),h_2(t),h_3(t);1)}\frac{\varphi_{t,\{1,2\}}(h_1(T),h_2(t),h_3(t);1)}{\varphi_{t,\{1,2\}}(h_1(t),h_2(t),h_3(t);1)}\\ &+\mathbf{1}_{\{\tau_2\le t,\tau_3> t\}}\frac{\varphi_{t,\{1,3\}}(h_1(t),h_2(\tau_2),h_3(t);1)}{\varphi_{t,\{1,3\}}(h_1(T),h_2(\tau_2),h_3(t);1)}\frac{\varphi_{t,\{1\}}(h_1(T),h_2(\tau_2),h_3(t);1)}{\varphi_{t,\{1\}}(h_1(t),h_2(\tau_2),h_3(t);1)}.\end{aligned}$$

Consequently, we conclude that the stochastic differential equation of the defaultable zero-coupon discount bond issued by obligor 1 for $n=3$ case is given by

$$\begin{aligned}dD^{(1)}_{tT} = D^{(1)}_{t-,T}\Bigg\{ &\Big(r_t+\lambda^{(1|\mathbb{G})}_t+\mathbf{1}_{\{\tau_2>t\}}\eta^{(2|\mathbb{G})}_{tT}+\mathbf{1}_{\{\tau_3>t\}}\eta^{(3|\mathbb{G})}_{tT}\Big)dt\\ &+\sigma_1\Sigma^{(1|\mathbb{G})}_{1:tT}dW^{(1|\mathbb{G})}_t+\mathbf{1}_{\{\tau_2>t\}}\sigma_2\Sigma^{(2|\mathbb{G})}_{1:tT}dW^{(2|\mathbb{G})}_t+\mathbf{1}_{\{\tau_3>t\}}\sigma_3\Sigma^{(3|\mathbb{G})}_{1:tT}dW^{(3|\mathbb{G})}_t\\ &-d\mathbf{1}_{\{\tau_1\le t\}}-\Big(1-\Xi^{(2|\mathbb{G})}_{1:tT}\Big)d\mathbf{1}_{\{\tau_2\le t\}}-\Big(1-\Xi^{(3|\mathbb{G})}_{1:tT}\Big)d\mathbf{1}_{\{\tau_3\le t\}}\Bigg\},\end{aligned}$$

with $D^{(1)}_{TT}=\mathbf{1}_{\{\tau_1>T\}}$.

References

Azizpour, S., Giesecke, K., and Schwenkler, G. (2018). Exploring the sources of default clustering. *Journal of Financial Economics* 129, 154-183.

Benzoni, L., Collin-Dufresne, P., Goldstein, R. S., and Helwege, J. (2015). Modeling credit contagion via the updating of fragile beliefs. Working paper available at https://ssrn.com/abstract=2016579.

Bielecki, T. R., Crépey, S., and Herbertsson, A. (2009). Markov chain models of portfolio credit risk. *Oxford Handbook of Credit Derivatives*, Springer, Berlin, Heidelberg.

Bielecki, T. R., and Rutkowski, M. (2002). *Credit Risk: Modeling, Valuation and Hedging*, Springer, Berlin, Heidelberg.

Bielecki, T. Vidozzi, A., and Vidozzi, L. (2008). A Markov copulae approach to pricing and hedging of credit index derivatives and rating-triggered step-up bonds. *Journal of Credit Risk* 4(1), 47-76.

Brody, D. C., Hughston, L. P., and Macrina, A. (2008). Information-based asset pricing. *International Journal of Theoretical and Applied Finance* 11, 107-142.

Brody, D. C., Hughston, L. P., and Macrina, A. (2010). Credit risk, market sentiment and randomly-timed default. *Stochastic Analysis 2010*. Springer, Berlin, Heidelberg, 267-280.

Brody, D. C., Hughston, L. P., and Macrina, A. (2011). Modelling information flows in financial markets. *Advanced Mathematical Methods for Finance*, Springer, Berlin, Heidelberg. 133-153.

Çetin, U., Jarrow, R., Protter, P., and Yildirim, Y. (2004). Modeling credit risk with partial information. *Annals of Applied Probability*, 14, 1167-1178.

Coculescu, D. (2017). A default system with overspiling contagion. Working paper available at https://ssrn.com/abstract=3041348.

Crépey, S., Jeanblanc, M., and Wu, D. L. (2013). Informationally dynamized Gaussian copula. *International Journal of Theoretical and Applied Finance*, 16(2), 1-29.

Crépey, S., and Song, S. (2017). Invariance properties in the dynamic Gaussian copula model. *ESAIM: Proceedings and Surveys*, 56, 22-41.

Das, S., Duffie, D., Kapadia, N., and Saita, L, (2007). Common failings: how corporate defaults are correlated. *Journal of Finance* 62(1), 93-117.

Davis, M., and Lo, V. (2001). Infectious defaults. *Quantitative Finance* 1, 382-387.

Duffie, D., Eckner, A., Horel, G., and Saita, L. (2009). Frailty correlated default. *Journal of Finance* 64(5), 2089-2123.

Duffie, D., and Lando, D. (2001). Term structures and credit spreads with incomplete accounting information. *Econometrica* 69, 633-664.

Duffie, D. and Singleton, K. (1999). Modeling term structure of defaultable bonds. *Review of Financial Studies* 12, 687-720.

Elliott, R., and Shen, J. (2015). Credit risk and contagion via self-exciting default intensity. *Annals of Finance* 11(3), 319-344.

Elouerkhaoui, Y. (2017). *Credit Correlation: Theory and Practice*. Palgrave Macmillan, Springer Nature, Switzerland.

Frey, R., and Backhaus, J. (2008). Pricing and hedging of portfolio credit derivatives with interacting default intensities. *International Journal of Theoretical and Applied Finance* 11(6), 611-634.

Frey, R., and Schmidt, T. (2012). Pricing and hedging of credit derivatives via the innovations approach to nonlinear filtering. *Finance and Stochastics* 16, 105-133.

Herbertsson, A. (2007). Pricing synthetic CDO tranches in a model with default contagion using the matrix-analytic approach. *Journal of Credit Risk* 4(4), 3-35.

Jarrow, R., and Protter, P. (2004). Structural versus reduced-form models: A new information based perspective. *Journal of Investment Management* 2(2), 34-43.

Jarrow, R., and Yu, F. (2001). Counterparty risk and the pricing of defaultable securities. *Journal of Finance* 56(5), 1765-1799.

El Karoui, N., Jeanblanc, M., and Jiao, Y. (2010). What happens after a default: The conditional density approach. *Stochastic Processes and their Applications* 120, 1011-1032.

El Karoui, N., Jeanblanc, M., and Jiao, Y. (2015). Density approach in modeling successive defaults. *SIAM Journal of Financial Mathematics* 6, 1-21.

Kusuoka, S. (1999). A remark on default risk models. *Advances in Mathematical Economics* 1, 69-82.

McNeil, A. J., Frey, R., and Embrechts, P. (2005). *Quantitative Risk Management, Concepts, Techniques and Tools.* Princeton University Press, Princeton and Oxford.

Nakagawa, H. (2001). A Filtering Model on Default Risk. *J. Math. Sci. Univ. Tokyo* 8, 107-142.

Quenez, M.-C., and Sulem, A. (2013). BSDEs with jumps, optimization and applications to dynamic risk measures. *Stochastic Processes and their Applications* 123, 3328-3357.

Revuz, D., and Yor, M. (1999). *Continuous Martingales and Brownian Motion 3rd. ed.*. Springer, Berlin, Heidelberg.

Schönbucher, P. J. (2003). *Credit Derivatives Pricing Models*, John Wiley & Sons, England.

Schönbucher, P. J., and Schubert, D. (2001). Copula-dependent defaults in intensity models. Preprint at ETH Zurich and Universität Bonn, available at https://ssrn.com/abstract=301968.

Yu, F. (2007). Correlated defaults in intensity-based models. *Mathematical Finance* 17(2), 155-173.

Yu, F. and Rutkowski, M. (2007). An Extension of the Brody-Hughston-Macrina Approach to Modeling of Defaultable Bonds. *International Journal of Theoretical and Applied Finance* 10(3), 557-589.

Zheng, H., and Jiang, L. (2009). Basket CDS pricing with interacting intensities. *Finance and Stochastics* 13, 445-469.