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Abstract

Counterparty risk remains at issue in over-the-counter derivative transactions after the financial crisis of 2008. While the margin for a derivative transaction can only be transfered until just before the counterparty's default, the exposure of the derivative transaction can vary stochastically during the margin period of risk (MPoR), that is, the period from the counterparty's default to the actual closed-out of the transaction, and so the anticipated positive exposure may not be recognized, which yields the counterparty risk. Since it is difficult to calculate the initial margin (IM) according to the regulations, it has been calculated in practice using a simplified method "ISDA SIMM". In this study, we derive an approximate formula on some indicators of counterparty risk for a stochastic volatility model and illustrate some numerical analyses for a call option in the SABR model as an example to examine the effect of discrepancy between regulations and practice in margin calculation. Our results imply that the IM calculated in practice may be insufficient as counterparty risk management, especially when the market is volatile.

Keywords: Counterparty risk; Initial margin; Margin period of risk; ISDA SIMM; SABR model

JEL Classification: C52, G13, G32.

1 Introduction

In derivatives transactions, contracts to transfer funds in the future, it is possible to calculate the present value of the amount to be transfered in the future at each point in time, and the transaction is usually closed out by paying (or receiving) the present value of the derivative. On the other hand, if the counterparty defaults within the contract period, you must pay the amount equivalent to the present value to the counterparty as contracted when the present value is negative from your side, while you may not be able to receive the full amount of the present value when it is positive from your side. Such a possibility of incurring losses in derivatives transactions due to the counterparty's default is recognized as the counterparty risk.

The counterparty risk has traditionally been considered in the context of risk management for individual financial institutions. After the collapse of Long Term Capital Management (LTCM) in 1998 and the global financial crisis in 2008, it has become mandatory for standard derivatives transactions to be cleared by a central counterparty (CCP). For centrally cleared trades, various margins are exchanged in accordance with the CCP's rules. On the other hand, even for the over-the-counter (OTC) derivatives transactions that are not subject to central clearing, reporting

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requirements on major transactions to Trade Repositories and the exchange of margin between the parties involved have been made mandatory. As a result, most transactions between major financial institutions are now subject to the margin exchanges, and their counterparty risks have been significantly reduced.

Generally, derivatives transactions are based on the ISDA Master Agreement established by the International Swaps and Derivatives Association (ISDA). Accordingly, it is common to conclude an additional clause named the Credit Support Annex (CSA) to the ISDA Master Agreement for margining.

Margin requirements include variation margin (VM), which preserves the exposure to counterparty at each point in time, and initial margin (IM), which preserves the variation of the exposure from counterparty's default to close out.

As such, it is natural to assume that the required VM is the value of the derivative transaction just before the counterparty's default, which is the last date of margin delivery. On the other hand, the required IM cannot be determined at the counterparty's default time because the value of derivative transactions can vary stochastically during the margin period of risk (MPoR), which is the period from the last date of margin delivery to the closed-out time of the transaction. Hence the regulation mandates that 99 % of the variation in positive exposure during the MPoR be preserved by the required IM. However, the IM calculations consistent with the regulations depend on the parameters to be input such as the volatility as well as the valuation model, the required IM is not always consistent among the parties.

This is likely to lead to a practical problem of making it hard to agree on the amount to be transfered with counterparties, so ISDA(2021) [9] developed the ISDA Standard Initial Margin Model (SIMM) as an easy-to-use IM calculation model as an industry standard.

The ISDA SIMM is intended to be simply tractable, and therefore does not elaborately reflect the features of complicated valuation models or changes in market conditions. Then it is possible that the IM requirement calculated with the SIMM may not be sufficient for preserving 99 % of the variation in positive exposure during the MPoR, which is required by the regulations. Thus in this study we aim to examine the effect on counterparty risk management of discrepancy between regulations and practice in margin calculation. More specifically, the purpose is to formulate and numerically analyze the discrepancy between the level of counterpart risk reduction during the MPoR targeted by the margin regulations and the IM requirement based on the ISDA SIMM used in practice.

First, we set a general model to mathematically discuss the margins for an OTC derivative transaction and define the potential future exposure (PFE) and expected positive exposure (EPE), often referred to as counterparty risk indicators. Also, we introduce the margin conservation ratio (Mratio) as an original indicator to see the extent to which the SIMM based IM meets that required by the regulations.

Then, for a generalized stochastic volatility model, which assumes that the value of a derivative transaction is given by the underlying asset and "pseudo volatility," we approximately derive the derivative value change over the MPoR so that we can achieve explicit approximation formulas of PFE, EPE, and Mratio for numerical analyses.

Furthermore, we illustrate some numerical analyses for a call option in the SABR model as an example. One of the results, in terms of Mratio, implies that the SIMM based IM in practice may be insufficient for counterparty risk management, especially when the market is volatile, while the SIMM IM works normally in low volatility cases.

2 Counterparty Risk and Margin Regulation

In this section, we introduce a general mathematical model to argue counterparty risk of derivatives trading. First, we introduce “potential future exposure (PFE)” and “expected positive exposure (EPE),” which are indicators commonly used in counterparty risk management, and formulate them in terms of “margin period of risk (MPoR)”, in other words, the difference between the margin to be posted in the event of counterparty’s default and the value of the derivative at the time when the derivative is actually closed out afterwards[1]. Next we pay attention to the difference in the calculation method of “initial margin (IM)” between regulations and practice, because such a difference indicates that the IM calculated by a simplified method used in practice may not be sufficient to achieve the risk reduction that is expected by the regulations.

2.1 General model for counterparty risk measurement

In this study, we first introduce a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ and every random variable and process representing derivatives, counterparty credit risk, and so on shall be defined on this probability space. We suppose that the probability measure $\mathbf{P}$ is a physical one.

We consider an OTC derivative contracted at time 0 with maturity T in the continuous time period $[0, T]$. Also, we introduce the filtration $(\mathcal{F}_t)_{t\in[0,T]}$, which corresponds to the information time flow available in the market.

We denote by $\{V(t)\}$ the $(\mathcal{F}_t)_{t\in[0,T]}$-adapted càdlàg (right-continuous with left limits) process representing the market value dynamics of the derivative. In short, if the derivative transaction is closed out at time t, you can receive $V(t)$ from the counterparty when $V(t) > 0$ (you are in a profitable position), while you have to pay $-V(t)$ to the counterparty when $V(t) < 0$ (you are in a lossy position).

Now, the important problem is that when $V(t) > 0$, if the counterparty falls into default, you may not fully receive the expected profit $V(t)$ and may incur some loss. This possibility of incurring losses due to the counterparty failure is the so-called “counterparty risk.”

Following the Pittsburgh Summit in 2009 (c.f. [10]) after the financial crisis happened in 2008, some regulations on derivatives transactions were introduced to avoid the emergence of counterparty risk. Many of the major derivatives traded among major financial institutions need to be cleared through the Central CounterParty (CCP) system. In such cases, the CCP calculates the margin requirement based on its own model and charges the necessary margin to each financial institution. CCP also settles and manages the margin requirements.

On the other hand, exotic derivatives transactions, which require complex system despite relatively low demand, and options products, for which there is no model to evaluate commonly accepted by all market participant.

In the case of OTC derivatives transactions between major financial institutions not involving CCPs, margin regulations require the exchange of margin in order to deal with counterparty risk. Anyway the margin is repaid to the counterparty if there is no problem when the derivative transaction is closed out. However, the posted margin can be used to compensate for the losses due to counterparty’s default if you are in a profitable position and the counterparty fails and cannot fully recover the exposure at time t. This is the basic principle of margin regulations on trading the OTC derivatives.

[1] According to BCBS(2019) [3], margin period of risk (MPoR) is defined by **the time period from the last exchange of collateral covering a netting set of transactions with a defaulting counterparty until that counterparty is closed out and the resulting market risk is re-hedged.**

Since there is no common system like CCP for OTC transactions, the method of exchanging margin is determined for each contract between the parties. Generally, derivatives transactions are executed on the "ISDA Master Agreement ([8])," which is the fundamental agreement established by the International Swaps and Derivatives Association (ISDA). Accordingly, the Credit Support Annex (CSA) is usually concluded for margin transfers as an annex to the ISDA Master Agreement.

As described above, it seems that the counterparty risk problem can be almost solved if the margin delivery is properly executed, but in reality, the derivatives are not liquidated immediately when the counterparty falls into default, but after the MPoR has passed.

Thus the following question arises: "Will the margin received at the time of counterparty's default be able to cover the profitable position of the derivative at the actual closed-out time?" In other words, when MPoR is given by a constant $\delta_{\mathrm{MPoR}}(>0)$[2], if the difference $V(t+\delta_{\mathrm{MPoR}})-M(t)$ is positive, it means that margins are not sufficient, that is, the counterparty risk may not be completely avoided.

In general, the required margin is divided into "variation margin (VM)" and "initial margin (IM)". The VM has the role of directly preserving the present value of derivatives, while the IM is introduced for the purpose of covering the variability of exposure during the margin period of risk (MPoR), which is the period between the time of the counterparty's default and the actual closed-out time of the transaction.

Since both VM and IM can randomly vary depending on the market value of the derivatives, they can be modeled by $(\mathcal{F}_t)$- predictable stochastic processes $\{\mathrm{VM}(t)\}$ and $\{\mathrm{IM}(t)\}$, respectively. Then the total necessary margin denoted by $\{M(t)\}$ can be given as $M(t)=\mathrm{VM}(t)+\mathrm{IM}(t)$.

As for the VM, the CSA usually stipulates that the required VM amount is always equal to the value of the derivative, so we suppose $\mathrm{VM}(t)=V(t-)$. The reason we use $V(t-)$ instead of $V(t)$ for VM is to account for the possibility of a jump in derivative value at the counterparty's default time. If the derivative value is not affected by the counterparty's default, we can assume $\mathrm{VM}(t)=V(t)$. Hereafter we assume $\mathrm{VM}(t)=V(t)$.

As a consequence, we have

$$\begin{aligned} V(t+\delta_{\mathrm{MPoR}})-M(t) &= V(t+\delta_{\mathrm{MPoR}})-V(t)+V(t)-\{\mathrm{VM}(t)+\mathrm{IM}(t)\} \\ &= V(t+\delta_{\mathrm{MPoR}})-V(t)-\mathrm{IM}(t). \end{aligned}$$

Andersen et al. [1] mentioned three kinds of modeling of the exposure during the MPoR:Classical+, Classical−, and Advanced. Our model corresponds to the Classical+, which is the simplest one.

The IM is intended to cover the variation in the value of derivative during the MPoR, but the required IM cannot be determined due to the uncertainty of the variation $V(t+\delta_{\mathrm{MPoR}})-V(t)$ at the time of counterparty's default, when the final margin is transfered, .

Regarding the problem, the regulations specify the required IM as the (conditional) 99 % point of $V(t+\delta_{\mathrm{MPoR}})-V(t)$, so that it can be evaulated at the time t of counterparty's default.

So the IM required by the regulations, denoted by $\mathrm{IM}^{\mathrm{Reg}}(t,\delta_{\mathrm{MPoR}})$, is defined by

$$\mathrm{IM}^{\mathrm{Reg}}(t,\delta_{\mathrm{MPoR}})=\inf\left\{y\in\mathbb{R}\mid \mathbf{P}\left(V(t+\delta_{\mathrm{MPoR}})-V(t)\geq y\mid\mathcal{F}_t\right)\leq 1\%\right\}. \tag{1}$$

For the purpose of specifically calculating the required IM by (1), it is necessary to identify the conditional probability distribution of the variation $V(t+\delta_{\mathrm{MPoR}})-V(t)$, hence the specific model that describes the dynamics of the process $\{V(t)\}$. However, valuation models for derivatives are

[2]Since MPoR is undetermined at the time of the counterparty's default, it seems natural to regard it as a random variable. However, closed-outs of derivatives transactions are often completed within a few days or two weeks at most after the counterparty's default, and so MPoR δ_{MPoR} is treated as a given positive constant in this study for ease of handling.

selected depending on the market environment, management policies of financial institutions, and progress in academic research and the specification of models for calculating the IM is not mentioned by the regulations, so the same model is not necessarily used among the financial institutions[3].

Therefore, because it is necessary in practice to calculate the required IM and transfer them smoothly without referring to the specific model of $\{V(t)\}$, a simplified calculation method for IM was developed mainly by ISDA and has become the industry standard as seen in the next subsection.

2.2 IM calculation used in practice: ISDA SIMM

Financial institutions usually have their own valuation models for each derivative transaction and calculate some sensitivities such as delta and vega for risk management. ISDA proposed a simplified method to calculate the IM using the sensitivities calculated by financial institutions and a common estimate of the market volatility. This is a feature of the industry standard IM calculation method called the ISDA Standard Initial Margin Model (SIMM). For simplicity, this study focuses on an interest rate option like a swaption and excludes margins due to concentration risk from the analysis.

Suppose that the market value of the derivative is given by a function of time t, the value of the underlying asset or variable, denoted by X, and the volatility of the underlying one, denoted by σ.

Then the formula for calculating the IM requirement $\mathrm{IM}^{\mathrm{CSA}}(t)$ of ISDA SIMM for interest rate derivatives and credit derivatives is as follows (refer to ISDA [9] for the detail)[4].

$$\mathrm{IM}^{\mathrm{CSA}}(t) = \mathrm{IM}^{\mathrm{Delta}}(t) + \mathrm{IM}^{\mathrm{Vega}}(t) + \mathrm{IM}^{\mathrm{Crtr}}(t), \tag{2}$$

where

$$\begin{aligned}
\mathrm{IM}^{\mathrm{Delta}}(t) &= \mathrm{RW} \cdot \{V(t, X(t) + 1\mathrm{bp}, \sigma(t)) - V(t, X(t), \sigma(t))\} \approx \mathrm{RW} \cdot \frac{\partial V}{\partial x}(t, X(t), \sigma(t)) \cdot 1\mathrm{bp}, \\
\mathrm{IM}^{\mathrm{Vega}}(t) &= \mathrm{VRW} \cdot \sigma(t) \cdot \{V(t, X(t), \sigma(t) + 1\mathrm{bp}) - V(t, X(t), \sigma(t))\} \\
&\approx \mathrm{VRW} \cdot \sigma(t) \cdot \frac{\partial V}{\partial \sigma}(t, X(t), \sigma(t)) \cdot 1\mathrm{bp}, \\
\mathrm{IM}^{\mathrm{Cvtr}}(t) &= \min\left\{1, \frac{14\mathrm{Days}}{\text{Time to maturity}}\right\} \sigma(t) \left(V(t, X(t), \sigma(t) + 1\mathrm{bp}) - V(t, X(t), \sigma(t))\right) \\
&\approx \min\left\{1, \frac{14\mathrm{Days}}{\text{Time to maturity}}\right\} \cdot \sigma(t) \cdot \frac{\partial V}{\partial \sigma}(t, X(t), \sigma(t)) \cdot 1\mathrm{bp}.
\end{aligned}$$

Here, by abuse of notation, we regard the value of the derivative as $V(t) = V(t, X(t), \sigma(t))$ with a function $V(t, x, \sigma)$ obtained from a specified valuation model. Also we suppose that we calculate the variations of $V(t, X(t), \sigma(t))$ for 1bp changes of the underlying asset $X(t)$ and the volatility $\sigma(t)$ as approximations of the Delta $\frac{\partial V}{\partial x}(t, X(t), \sigma(t))$ and the Vega $\frac{\partial V}{\partial \sigma}(t, X(t), \sigma(t))$.

Also, RW and VRW are positive constants called the SIMM coefficients, which are supposed to correspond to the 99 % of the small variations for a small change of the underlying asset and the volatility, respectively. ISDA updates the SIMM coefficients annually by estimating them with the market data for the last three years and for the stressed one year of increasing volatility such as the year 2008 in order to keep the values conservative[5].

[3] For example, LCH, a British clearing house group, has announced that it calculates its initial margin requirements for the swap clearing market using VaR or Expected Shortfall, which is derived from historical simulations of five-day fluctuations over the past 10 years. See the website https://www.lch.com/risk-management/risk-management-ltd (Last accessed on July 6, 2023).

[4] As for derivatives other than interest rate derivatives and credit derivatives, the basic concept of calculating IM requirements remains the same, although the formulas are different.

[5] Such a parameter estimation is often called "3+1 method."

We remark that MPoR is not explicitly included in the ISDA SIMM formula for calculating the required IM because the SIMM coefficients are estimated under the presumption that MPoR is set as 10 days.

2.3 Indicators of counterparty risk: PFE and EPE

Gregory [6] defines the **potential future exposure (PFE)** as an indicator of counterparty risk by *the level at which the probability that the positive risk exposure exceeds the amount of the counterparty risk by no more than 1% under a real probability measure*. He also does the **expected positive exposure (EPE)** by *the expected value of the positive risk exposure*. As you notice, the PFE corresponds to Value at Risk (VaR) with 99% confidence level, which is often used as a market risk measure.

Thus we specify the PFE and EPE for MPoR δ_{MPoR} at time t as $\mathcal{F}_t$-conditional random variables as follows.

$$\begin{aligned}
\mathrm{PFE}(t;\delta_{\mathrm{MPoR}}) &= \mathrm{ess.inf}\left\{y \in \mathbb{R} \mid \mathbf{P}\left((V(t+\delta_{\mathrm{MPoR}}) - M(t))^+ \geq y \mid \mathcal{F}_t\right) \leq 0.01\right\} \qquad (3)\\
&= \mathrm{ess.inf}\left\{y \in \mathbb{R} \mid \mathbf{P}\left((V(t+\delta_{\mathrm{MPoR}}) - V(t) - \mathrm{IM}(t))^+ \geq y \mid \mathcal{F}_t\right) \leq 0.01\right\},\\
\mathrm{EPE}(t;\delta_{\mathrm{MPoR}}) &= \mathbf{E}^{\mathbf{P}}\left[(V(t+\delta_{\mathrm{MPoR}}) - M(t))^+ \mid \mathcal{F}_t\right] \qquad (4)\\
&= \mathbf{E}^{\mathbf{P}}\left[(V(t+\delta_{\mathrm{MPoR}}) - V(t) - \mathrm{IM}(t))^+ \mid \mathcal{F}_t\right],
\end{aligned}$$

where $a^+ := \max\{a, 0\}$ for $a \in \mathbb{R}$.

Note that we focus on the **Positive Exposure (PE)** $(V(t+\delta_{\mathrm{MPoR}}) - M(t))^+$ because the excess of the liquidated amount after MPoR of the derivative over the margin preserved is at issue. It follows that the theoretical PFE for the regulation based $\mathrm{IM}^{\mathrm{Reg}}(t, \delta_{\mathrm{MPoR}})$ given as (1) becomes zero.

On the other hand, if we calculate PE in practice based on the ISDA SIMM, we see that the practical PFE denoted by $\mathrm{PFE}^{\mathrm{CSA}}$ does not vanish unless $\mathrm{IM}^{\mathrm{Reg}}(t, \delta_{\mathrm{MPoR}}) \leq \mathrm{IM}^{\mathrm{CSA}}(t)$ as below.

$$\begin{aligned}
&\mathrm{PFE}^{\mathrm{CSA}}(t;\delta_{\mathrm{MPoR}})\\
&= \mathrm{ess.inf}\left\{y \in \mathbb{R} \mid \mathbf{P}\left(\left(V(t+\delta_{\mathrm{MPoR}}) - V(t) - \mathrm{IM}^{\mathrm{CSA}}(t)\right)^+ \geq y \mid \mathcal{F}_t\right) \leq 0.01\right\}\\
&= \left(\mathrm{ess.inf}\left\{y \in \mathbb{R} \mid \mathbf{P}\left((V(t+\delta_{\mathrm{MPoR}}) - V(t))^+ \geq y \mid \mathcal{F}_t\right) \leq 0.01\right\} - \mathrm{IM}^{\mathrm{CSA}}(t)\right)^+\\
&= \left(\mathrm{IM}^{\mathrm{Reg}}(t, \delta_{\mathrm{MPoR}}) - \mathrm{IM}^{\mathrm{CSA}}(t)\right)^+.
\end{aligned}$$

Since counterparty risk is often used to visualize the impact on a financial institution's capital and the value of derivatives transactions, it is generally measured in terms of monetary amounts like PFE and EPE. In contrast, this study focuses on the extent to which the IM requirement in practice meets that by the regulations, which is specified as the conservation ratio of 99 % of the exposure variation during MPoR. Hence, in order to see the extent, we define the margin conservation ratio (Mratio) at time t, which is an original indicator of this study, by the $\mathcal{F}_t$-conditional probability that the positive exposure in practice equals zero like

$$\begin{aligned}
\mathrm{Mratio}(t;\delta_{\mathrm{MPoR}}) &= \mathbf{P}\left(\left(V(t+\delta_{\mathrm{MPoR}}) - V(t) - \mathrm{IM}^{\mathrm{CSA}}(t)\right)^+ = 0 \mid \mathcal{F}_t\right) \qquad (5)\\
&= \mathbf{P}\left(V(t+\delta_{\mathrm{MPoR}}) - V(t) \leq \mathrm{IM}^{\mathrm{CSA}}(t) \mid \mathcal{F}_t\right).
\end{aligned}$$

It follows from the definition of $\mathrm{IM}^{\mathrm{Reg}}(t;\delta_{\mathrm{MPoR}})$ given in (1) that the Mratio is 99% when $\mathrm{IM}^{\mathrm{CSA}}(t) = \mathrm{IM}^{\mathrm{Reg}}(t;\delta_{\mathrm{MPoR}})$.

Anyhow that is why the discrepancy between regulations and practice in IM calculation can be a problem in counterparty risk management in OTC derivatives transactions even with initial margin.

In the next section, we show a specific formulas for PFE, EPE and Mratio in practice for a general stochastic volatility model. Especially we prepare a numerical analysis for the SABR model with a European call option.

3 Counterparty Risk Measurement Specified for a Stochastic Volatility Model

In the previous section, we discuss the possibility that the IM requirements calculated with the ISDA SIMM used in practice may not reach the level required by the regulations. In this section, we firstly introduce a generalized stochastic volatility model and theoretically derive an approximate evaluation formula for PE in practice by using Ito's formula. Based on the approximation of PE, we show how to calculate PFE, EPE, and Mratio presented in the previous section. Furthermore, as an illustrative example, we present how to obtain the PFE and EPE of a payer swaption, which is a type of interest rate derivatives that can be regarded as a European call option with a forward swap rate as the underlying asset, under the SABR model, which is one of the stochastic volatility models often used in practice.

3.1 Derivation of approximate counterparty risk indicators

Hereafter we suppose that the value process $\{V(t)\}$ of a derivative transaction can be expressed by $V(t) = V(t, X(t), \sigma(t))$ as a function of time t, the underlying asset $X(t)$ and the volatility of the underlying asset (or something that plays a similar role) $\sigma(t)$ so as to be consistent with the ISDA SIMM that uses the Vega, the sensitivity of the derivative value to the volatility.

We assume that the underlying asset process $\{X(t)\}$ and the pseudo volatility process $\{\sigma(t)\}$ satisfy the following stochastic differential equations (SDEs) called a generalized stochastic volatility model. We note that $\sigma(t)$ itself is not the true volatility of the underlying asset, but in most specific models it essentially plays a role in driving the volatility, so we call it the pseudo volatility.

$$dX(t) = \mu_X(t, X(t), \sigma(t))dt + \sigma_X(t, X(t), \sigma(t))dW_X^{\mathbf{P}}(t), \tag{6}$$

$$d\sigma(t) = \mu_\sigma(t, X(t), \sigma(t))dt + \sigma_\sigma(t, X(t), \sigma(t))dW_\sigma^{\mathbf{P}}(t), \tag{7}$$

where $\mu_X(t, x, \sigma), \mu_\sigma(t, x, \sigma), \sigma_X(t, x, \sigma)$ and $\sigma_\sigma(t, x, \sigma)$ are "good" functions satisfying some regularity conditions that guarantee the existence and uniqueness of strong solutions for the above SDEs, and $W_X^{\mathbf{P}}(t)$ and $W_\sigma^{\mathbf{P}}(t)$ are correlated standard Brownian motions under the physical measure $\mathbf{P}$ with constant correlation $\rho \in [-1, 1]$, that is, $dW_X^{\mathbf{P}}(t)dW_\sigma^{\mathbf{P}}(t) = \rho dt$. .

From Ito's formula, it follows that if the function $V(t, x, \sigma)$ is a $C^{1,2,2}$-function, we have (for simplicity, write $\partial_x V(t)$ and $\partial^2_{xx} V(t)$ instead of $\dfrac{\partial V}{\partial x}(t, X(t), \sigma(t))$ and $\dfrac{\partial^2 V}{\partial x^2}(t, X(t), \sigma(t))$ respectively, and so on)

$$
\begin{aligned}
dV(t) &= dV(t, X(t), \sigma(t)) \\
&= \partial_t V(t)dt + \partial_x V(t)dX(t) + \partial_\sigma V(t)d\sigma(t) \\
&\quad + \frac{1}{2}\partial^2_{xx}V(t)d\langle X\rangle(t) + \frac{1}{2}\partial^2_{\sigma\sigma}V(t)d\langle\sigma\rangle(t) + \partial^2_{x\sigma}V(t)d\langle X,\sigma\rangle(t) \\
&= \partial_x V(t)\sigma_X(t, X(t), \sigma(t))dW^{\mathbf{P}}_X(t) + \partial_\sigma V(t)\sigma_\sigma(t, X(t), \sigma(t))dW^{\mathbf{P}}_\sigma(t) \\
&\quad + \Big\{\partial_t V(t) + \partial_x V(t)\mu_X(t, X(t), \sigma(t)) + \partial_\sigma V(t)\mu_\sigma(t, X(t), \sigma(t)) \\
&\qquad + \frac{1}{2}\partial^2_{xx}V(t)\sigma_X(t, X(t), \sigma(t))^2 + \frac{1}{2}\partial^2_{\sigma\sigma}V(t)\sigma_\sigma(t, X(t), \sigma(t))^2 \\
&\qquad + \partial^2_{x\sigma}V(t)\rho\ \sigma_X(t, X(t), \sigma(t))\sigma_\sigma(t, X(t), \sigma(t))\Big\}dt.
\end{aligned} \tag{8}
$$

In the remainder of this subsection, we make time t fixed.

Since MPoR δ_{MPoR} can be regarded as a sufficiently short period like a few days or two weeks at most ($\frac{14\mathrm{Days}}{365\mathrm{Days}}$ is used in the numerical illustration later), it seems possible to apply the discrete approximation of (8) by viewing $dt \approx \delta_{\mathrm{MPoR}}$. Accordingly, $dW_X(t)$ and $dW_\sigma(t)$ can be approximately replaced with $\sqrt{\delta_{\mathrm{MPoR}}}Z_X$ and $\sqrt{\delta_{\mathrm{MPoR}}}Z_\sigma$, where Z_X and Z_σ are two standard normally distributed random variables with correlation ρ.

Then PE based on the ISDA SIMM given by $\big(V(t+\delta_{\mathrm{MPoR}}) - V(t) - \mathrm{IM}^{\mathrm{CSA}}(t)\big)^+$ can be represented as follows.

$$
\begin{aligned}
&V(t+\delta_{\mathrm{MPoR}}) - V(t) - \mathrm{IM}^{\mathrm{CSA}}(t) \\
&\quad \approx \partial_x V(t)\sigma_X(t, X(t), \sigma(t))\sqrt{\delta_{\mathrm{MPoR}}}Z_X + \partial_\sigma V(t)\sigma_\sigma(t, X(t), \sigma(t))\sqrt{\delta_{\mathrm{MPoR}}}Z_\sigma \\
&\quad + \Big(\partial_x V(t)\mu_X(t, X(t), \sigma(t)) + \partial_\sigma V(t)\mu_\sigma(t, X(t), \sigma(t)) \\
&\quad + \partial_t V(t) + \frac{1}{2}\partial^2_{xx}V(t)\sigma_X(t, X(t), \sigma(t))^2 + \frac{1}{2}\partial^2_{\sigma\sigma}V(t)\sigma_\sigma(t, X(t), \sigma(t))^2 \\
&\quad + \partial^2_{x\sigma}V(t)\rho\sigma_X(t, X(t), \sigma(t))\sigma_\sigma(t, X(t), \sigma(t))\Big)\delta_{\mathrm{MPoR}} - \mathrm{IM}^{\mathrm{CSA}}(t).
\end{aligned} \tag{9}
$$

Here we define $\mathcal{F}_t$-measurable random variables A_X, A_σ, and A_C by

$$
A_X(t) := \partial_x V(t)\sigma_X(t, X(t), \sigma(t))\sqrt{\delta_{\mathrm{MPoR}}} \tag{10}
$$

$$
A_\sigma(t) := \partial_\sigma V(t)\sigma_\sigma(t, X(t), \sigma(t))\sqrt{\delta_{\mathrm{MPoR}}} \tag{11}
$$

$$
\begin{aligned}
A_C(t) := \Big\{&\partial_x V(t)\mu_X(t, X(t), \sigma(t)) + \partial_\sigma V(t)\mu_\sigma(t, X(t), \sigma(t)) \\
&+ \partial_t V(t) + \frac{1}{2}\partial^2_{xx}V(t)\sigma_X(t, X(t), \sigma(t))^2 + \frac{1}{2}\partial^2_{\sigma\sigma}V(t)\sigma_\sigma(t, X(t), \sigma(t))^2 \\
&+ \partial^2_{x\sigma}V(t)\rho\sigma_X(t, X(t), \sigma(t))\sigma_\sigma(t, X(t), \sigma(t))\Big\}\delta_{\mathrm{MPoR}} - \mathrm{IM}^{\mathrm{CSA}}(t).
\end{aligned} \tag{12}
$$

As a consequence, we have

$$
V(t+\delta_{\mathrm{MPoR}}) - V(t) - \mathrm{IM}^{\mathrm{CSA}}(t) \approx A_X(t)Z_X + A_\sigma(t)Z_\sigma + A_C(t).
$$

Since Z_X and Z_σ are independent of $\mathcal{F}_t$ and each one follows a standard normal distribution, we see the conditional expectation and the conditional variance of $A_X(t)Z_X + A_\sigma(t)Z_\sigma + A_C(t)$ are represented in the followings.

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}}\left[A_X(t)Z_X + A_\sigma(t)Z_\sigma + A_C(t) \mid \mathcal{F}_t\right] = A_C(t),\\
&\mathbf{Var}^{\mathbf{P}}\left[A_X(t)Z_X + A_\sigma(t)Z_\sigma + A_C(t) \mid \mathcal{F}_t\right] = A_X(t)^2 + 2\rho A_X(t)A\sigma(t) + A_\sigma(t)^2.
\end{aligned}$$

Moreover, letting $A_z(t) := \sqrt{A_X(t)^2 + 2\rho A_X(t)A\sigma(t) + A_\sigma(t)^2}$, we can obtain another standard normal variable Z independent of $\mathcal{F}_t$ specified by

$$Z := \frac{A_X(t)Z_X + A_\sigma(t)Z_\sigma}{A_z(t)}.$$

Thus we have

$$V(t+\delta_{\mathrm{MPoR}}) - V(t) - \mathrm{IM}^{\mathrm{CSA}}(t) \approx A_z(t)Z + A_C(t).$$

Finally it follows from (3) and (4) with the above argument that $\mathrm{PFE}^{\mathrm{CSA}}(t;\delta_{\mathrm{MPoR}})$ and $\mathrm{EPE}^{\mathrm{CSA}}(t;\delta_{\mathrm{MPoR}})$ can be obtained as below.

$$\begin{aligned}
\mathrm{PFE}^{\mathrm{CSA}}(t;\delta_{\mathrm{MPoR}}) &\approx \left(\Phi^{-1}(0.99)A_z(t) + A_C(t)\right)^+\\
&= (2.33A_z(t) + A_C(t))^+, \qquad (13)\\
\mathrm{EPE}^{\mathrm{CSA}}(t;\delta_{\mathrm{MPoR}}) &\approx \mathbf{E}^{\mathbf{P}}\left[(A_z(t)Z + A_C(t))^+ \mid \mathcal{F}_t\right]\\
&= A_z(t)\mathbf{E}^{\mathbf{P}}\left[Z\mathbf{1}_{\left\{Z>-\frac{A_C(t)}{A_z(t)}\right\}} \mid \mathcal{F}_t\right] + A_C(t)\mathbf{E}^{\mathbf{P}}\left[\mathbf{1}_{\left\{Z>-\frac{A_C(t)}{A_z(t)}\right\}} \mid \mathcal{F}_t\right]\\
&= A_z(t)\int_{-\frac{A_C(t)}{A_z(t)}}^{+\infty} z\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}\,dz + A_C(t)\Phi\left(\frac{A_C(t)}{A_z(t)}\right)\\
&= \frac{A_z(t)}{\sqrt{2\pi}}\exp\left(-\frac{A_C(t)^2}{2A_z(t)^2}\right) + A_C(t)\Phi\left(\frac{A_C(t)}{A_z(t)}\right), \qquad (14)
\end{aligned}$$

where $\mathbf{1}_A$ is the indicator variable of the event $A \in \mathcal{F}$ and $\Phi(\cdot)$ is the standard normal distribution function.

Moreover the margin conservation ratio (Mratio) defined in (5) can be calculated by

$$\begin{aligned}
\mathrm{Mratio}(t) &\approx \mathbf{P}\left((A_z(t)Z + A_C(t))^+ = 0 \mid \mathcal{F}_t\right)\\
&= \mathbf{P}\left(A_z(t)Z + A_C(t) \le 0 \mid \mathcal{F}_t\right)\\
&= \mathbf{P}\left(Z \le -\frac{A_C(t)}{A_z(t)} \mid \mathcal{F}_t\right) = \Phi\left(-\frac{A_C(t)}{A_z(t)}\right). \qquad (15)
\end{aligned}$$

3.2 Example: payer swaption with SABR model

For the numerical analysis of counterparty risk indicators in the next section, we set up a derivatives transaction and its valuation model to be used as an example.

As mentioned above, many derivatives transactions have become subject to centralized clearing, but exotic derivatives with small trading volumes and some products for which valuation models are difficult to construct are still traded over-the-counter. In this study, we will use a "payer swaption," an interest rate derivative that is commonly traded over-the-counter despite its relatively high trading volume, as an example for numerical analysis from a practical viewpoint.

This option has the forward swap rate as the underlying asset and can be exercised only on the expire date. When the option is exercised at the maturity, a swap transaction on which the swaption holder pays the fixed rate and receives the floating rate can be initiated (or cleared with exchanging cash, depending on the contract) at a strike rate K. So if $X(t)$ stands for the forward swap rate at time t, the option payoff $V(T)$ at maturity T is given by that of the so-called European call option as follows.

$$V(T) = (X(T) - K)^+ . \tag{16}$$

Next, as a valuation model, we adopt the SABR model, a kind of stochastic volatility models proposed by Hagan et al.[7]. The SABR model is often used in interest rate derivatives transactions since it has the feature that volatility skew can be taken into account. The SABR model is generally formulated as the underlying forward rate $\{X(t)\}$ and the pseudo volatility $\{\sigma(t)\}$ follow the following SDEs under a risk-neutral probability measure $\mathbf{Q}$.

$$dX(t) = \sigma(t) \left(X(t)\right)^{\beta} dW_X^{\mathbf{Q}}(t), \tag{17}$$

$$d\sigma(t) = \nu\sigma(t)dW_\sigma^{\mathbf{Q}}(t), \tag{18}$$

where $\{W_X^{\mathbf{Q}}(t)\}$ and $\{W_\sigma^{\mathbf{Q}}(t)\}$ are standard Brownian motions under the measure $\mathbf{Q}$ with $dW_X^{\mathbf{Q}}(t)dW_\sigma^{\mathbf{Q}}(t) = \rho dt$, and the parameters $\sigma(0) > 0, \beta \in [0, 1], \nu > 0$ and $\rho \in [-1, 1]$ are constants.

Note that we can view the SABR model as an example of the generalized stochastic volatility model specified by (6) and (7) by setting $\mu_X(t, x, \sigma) = \mu_\sigma(t, x, \sigma) = 0, \sigma_X(t, x, \sigma) = \sigma x^\beta$ and $\sigma_\sigma(t, x, \sigma) = \nu\sigma$, though the underlying probability measures are different.

In order to manage counterparty risk in PFE and so on, it is necessary to transform the dynamics under the risk-neutral measure $\mathbf{Q}$ to that under the physical measure $\mathbf{P}$, which will be discussed later.

Hagan et al. [7] derived two types of approximations for pricing the European call option under the SABR model. One is consistent with the case of $\beta = 1$, or the Black model (the forward rate$X(t)$ is log-normal), and the other is with $\beta = 0$, or the Bachelier model ($X(t)$ is normal). Although negative interest rates have been observed in developed countries since the beginning of the 2010s, the Black (lognormal) model cannot allow for negative interest rates, so this study use the approximation pricing formula consistent with the Bachelier (normal) model. Let σ_N be the pseudo volatility in terms of the Bachelier's model with $dX(t) = \sigma_N dW_X^{\mathbf{Q}}(t)$.

Denote by $V(t) = V(T - t, X(t), \sigma_N(t), K)$ the price at time $t \in [0, T]$ of the European call option with strike K and maturity T. Here we suppose that $\sigma_N(t)$ is the implied pseudo volatility at time t given as below, which is consistent with the Bachelier model with $dX(t) = \sigma_N dW_X^{\mathbf{Q}}(t)$.

It follows from Appendix B.2. in Hagan et al. [7] that we can approximate $V(T-t, X(t), \sigma_N(t), K)$ for $\beta \in (0, 1)$ as follows.

$$V(T - t, X(t), \sigma_N(t), K) \approx (X(t) - K)\, \Phi\left(d(t)\right) + \sigma_N(t)\sqrt{T - t} \cdot \phi\left(d(t)\right), \tag{19}$$

where

$$z(t) := z(X(t), \sigma(t), \beta, \nu, K) = \frac{\nu}{\sigma(t)} \frac{X(t)^{1-\beta} - K^{1-\beta}}{1-\beta},$$

$$\mathfrak{X}(z, \rho) = \log\left(\frac{\sqrt{1 - 2\rho z + z^2} - \rho + z}{1-\rho}\right),$$

$$\sigma_N(t) := \sigma_N(T-t, X(t), \sigma(t), \beta, \rho, \nu, K) = \frac{\sigma(t)(1-\beta)(X(t)-K)}{X(t)^{1-\beta} - K^{1-\beta}} \frac{z(t)}{\mathfrak{X}(z(t), \rho)}$$
$$\left\{1 + \left[(-\frac{1}{24}\frac{\beta(2-\beta)(1-\beta)^2\sigma(t)^2(\log\frac{X(t)}{K})^2}{(X(t)^{1-\beta} - K^{1-\beta})^2} + \frac{\rho\nu\sigma(t)}{4}\frac{X(t)^\beta - K^\beta}{X(t)-K} + \frac{2-3\rho^2}{24}\nu^2\right](T-t)\right\},$$

$$d(t) := d(T-t, X(t), \sigma_N(t), K) = \frac{X(t) - K}{\sigma_N(t)\sqrt{T-t}}.$$

Using the approximate analytical solution for the option price, we can calculate the partial derivatives such as $\frac{\partial V}{\partial x}(t)$ and $\frac{\partial V}{\partial \sigma}(t)$ for obtaining the counterparty risk indicators.

We suppose that the risk premiums $\theta_X(t)$ and $\theta_\sigma(t)$ are given as $(\mathcal{F}_t)$-adapted processes satisfy

$$dW_X^{\mathbf{P}}(t) = dW_X^{\mathbf{Q}}(t) - \theta_X(t)dt,$$
$$dW_\sigma^{\mathbf{P}}(t) = dW_\sigma^{\mathbf{Q}}(t) - \theta_\sigma(t)dt.$$

Thus it follows from the Girsanov-Maruyama theorem that Brownian motions $W_X^{\mathbf{Q}}(t)$ and $W_\sigma^{\mathbf{Q}}(t)$ under the risk-neutral measure $\mathbf{Q}$ is converted into Brownian motions $W_X^{\mathbf{P}}(t)$ and $W_\sigma^{\mathbf{P}}(t)$ under the physical measure $\mathbf{P}$, so we can obtain the dynamics of $X(t)$ and $\sigma(t)$ with $W_X^{\mathbf{P}}(t)$ and $W_\sigma^{\mathbf{P}}(t)$ under the measure $\mathbf{P}$.

Since we have

$$\mu_X(t) = \theta_X(t)\sigma(t)\left(X(t)\right)^\beta,$$
$$\mu_\sigma(t) = \theta_\sigma(t)\nu\sigma(t),$$

we can obtain the $\mathcal{F}_t$-measurable components $A_X(t), A_\sigma(t)$, and $A_C(t)$ (given in (10), (11), and (12)) needed to calculate the risk indicators PFE, EPE, and Mratio for MPoR θ (given in (13), (14), together with $A_z(t) := \sqrt{A_X(t)^2 + 2\rho A_X(t)A\sigma(t) + A_\sigma(t)^2}$).

$$A_X(t) = \left(\Phi(d(t)) + \sqrt{T-t}\phi(d(t))\frac{\partial\sigma_N}{\partial x}(t)\right)\sigma(t)X(t)^\beta\sqrt{\delta_{\text{MPoR}}} \tag{20}$$

$$A_\sigma(t) = \sqrt{T-t}\phi(d(t))\frac{\partial\sigma_N}{\partial\sigma}(t)\nu\sigma(t)\sqrt{\delta_{\text{MPoR}}} \tag{21}$$

$$\begin{aligned} A_C(t) = &\left[\left(\Phi(d(t)) + \sqrt{T-t}\phi(d(t))\frac{\partial\sigma_N}{\partial x}(t)\right)\sigma(t)X(t)^\beta\theta_X(t)\right. \\ &+ \sqrt{T-t}\phi(d(t))\frac{\partial\sigma_N}{\partial\sigma}(t)\nu\sigma(t)\theta_\sigma(t) + \left(\sqrt{T-t}\frac{\partial\sigma_N}{\partial t}(t) - \frac{\sigma_N(t)}{2\sqrt{T-t}}\right)\phi(d(t)) \\ &\left.+ \left(\frac{1}{2}\partial_{xx}^2V(t)X(t)^{2\beta} + \frac{1}{2}\partial_{\sigma\sigma}^2V(t)\nu^2 + \partial_{x\sigma}^2V(t)\rho\nu X(t)^\beta\right)\sigma(t)^2\right]\cdot\delta_{\text{MPoR}} \\ &- \left[\text{RW}\left(\Phi(d(t)) + \sqrt{T-t}\phi(d(t))\frac{\partial\sigma_N}{\partial x}(t)\right) + (\text{VRW} + \text{SF})\,\sigma(t)\sqrt{T-t}\phi(d(t))\frac{\partial\sigma_N}{\partial\sigma}(t)\right], \end{aligned} \tag{22}$$

where RW and VRW are the SIMM coefficients and SF $:= \min\left\{1, \frac{14\text{Days}}{T-t}\right\}$ introduced in subsection 2.2.

We discuss in section A of Appendix how to derive the above Greeks for the pricing formula (19) of European call option under the SABR model.

We can further obtain specific expressions for the partial derivatives of σ_N with respect to the underlying and the pseudo volatility, but since they would be very complicated expressions, the partial derivatives will be numerically computed by taking central difference.

4 Numerical illustration

In this section, as prepared in the last subsection, we numerically calculate the counterparty risk indicators such as PFE under some conditions for the payer swaption for the SABR model to see how different parameters like the pseudo volatility affect the indicators. We also attempt to see the source of the PFE by decomposing the exposure into three components related to Delta, Vega, and Curvature.

4.1 Assumptions for Numerical Calculations for Counterparty Risk Indicators

First, we assume that the payer swaption used in the numerical analysis has the strike price of $K = 3.00\%$ and the time to maturity of $T - t = 1$ (Year), that is, the payoff at maturity is given by

$$V(T) = (X(T) - 3.00\%)^+ .$$

For the coefficients of the SABR model, the risk premiums $\theta_X(t)$ and $\theta_\sigma(t)$ are assumed to be zero, partly because of the difficulty in estimating them. Equivalently, both $X(t)$ and $\sigma(t)$ are assumed to have the same dynamics under $\mathbf{P}$ and $\mathbf{Q}$. In addition, given the situation where, at the time of writing this paper, the volatility is increasing as the interest rates rise, we suppose $\beta = 0.75$ so that the forward rate becomes closer to the log-normal model with $\beta = 1$. Furthermore, we suppose $\nu = 0.30$ for the volatility of the pseudo volatility and $\rho = 0.5$ for the correlation between the underlying asset and its pseudo volatility. In short, we suppose

$$dX(t) = \sigma(t)X(t)^{0.75}dW_X^{\mathbf{P}}(t), d\sigma(t) = 0.30\sigma(t)dW_\sigma^{\mathbf{P}}(t), dW_X^{\mathbf{P}}(t)dW_\sigma^{\mathbf{P}}(t) = 0.50dt.$$

The ISDA SIMM based IM, denoted by IM^{CSA}, depends upon the SIMM coefficients RW and VRW published by ISDA. The SIMM coefficients for interest rate derivatives in 2021 published by ISDA [9] are shown in Table 1.

The RWs for regular currencies[6] has generally remain between 50 and 60, except for the periods of less than one year. The VRW is 0.18 for all currencies. In this study, we suppose RW= 60 and VRW= 0.20 for all the periods to simplify the calculations.

The MPoR is set to 10 business days, namely

$$\delta_{\text{MPoR}} = \frac{10 \text{ Business Days}}{1 \text{ Year}} \approx \frac{14 \text{ Days}}{365 \text{ Days}},$$

according to the BCBS [2].

In summary, the parameters and other factors assumed in the numerical calculations are as shown in Table2.

[6]The reguar currencies are USD, EUR, GBP, CHF, AUD, NZD, CAD, SEK, NOK, DKK, HKD, KRW, SGD, and TWD, while the only low-vol currency designated is the JPY. Other currencies are classified as high-vol currencies.

Table 1: The SIMM coefficients revised in 2021

type	ccy	2w	1m	3m	6m	1yr	2yr	3yr	5yr	10yr	15yr	20yr	30yr
RW	regular	114	106	95	74	66	61	56	52	53	57	60	66
	low-vol	15	18	8.6	11	13	15	18	20	19	19	20	23
VRW	All	0.18											

Table 2: The parameters and other factors assumed in the numerical calculations

	parameter	value
Option	K	3%
	$T-t$	1 Year
Model	μ_X, μ_σ	0
	β	0.75
	ν	30%
	ρ	0.5
SIMM	RW	50
	VRW	0.2
MPoR	δ_{MPoR}	14 / 365

4.2 Numerical results on counterparty risk indicators

In the following, we numerically calculate the counterparty risk indicators of PFE, EPE, and Mratio based on the ISDA SIMM method for the values of $X(t)$ ranging from 1% to 7%, for three cases of the pseudo volatility: for cases of $\sigma(t)$ = 10%, 15% and 20%.

The results are shown in Figure 1. We can easily check whether the IM in practice meets the requirements of the regulation by seeing the Mratio in the graphs. Specifically, we can see that in the low pseudo volatility with $\sigma(t)$ =10%, IM^{CSA} is sufficient even for the deep in-the-money, or the high underlying forward rate. It is particularly noticeable that especially for the high pseudo volatility case of $\sigma(t)$ =20%, the PFE (in red line) rapidly increases while the Mratio (in blue line) rapidly decreases when the underlying forward rate $X(t)$ is in-the-money, or greater than the strike price $K = 3.00\%$.

Next, we also calculate the counterparty risk indicators for the values of $\sigma(t)$ ranging from 5% to 25% for the out-of-the money case with $X(t) = 2\%$ and for the in-the-money case with $X(t) = 4\%$ and 6%, respectively.

The results are shown in Figure 2. It can be seen that in the out-of-the money case, IM^{CSA} is sufficient even when the volatility is high, but as the degree of in-the-money increases, the decrease in Mratio becomes steeper with increase in the pseudo volatility.

4.3 Factor decomposition of risk

In the ISDA SIMM, as shown in (2), the total IM^{CSA} is calculated by summing IM^{Delta}, IM^{Vega} and IM^{Cvtr}. If it is possible to see the extent to which the PFE is related to Delta, Vega, and Curvature factors, this will be useful for counterparty risk management.

Remeber that we have

$$V(t+\delta_{\text{MPoR}}) - V(t) \approx A_X(t)Z_X + A_\sigma(t)Z_\sigma + A_C(t),$$

where Z_X and Z_σ are standard normal random variables. In light of this, it follows from (9) - (12) that the exposure during the MPoR over the SIMM based IM, $V(t+\delta_{\text{MPoR}}) - V(t) - \text{IM}^{\text{CSA}}(t)$,

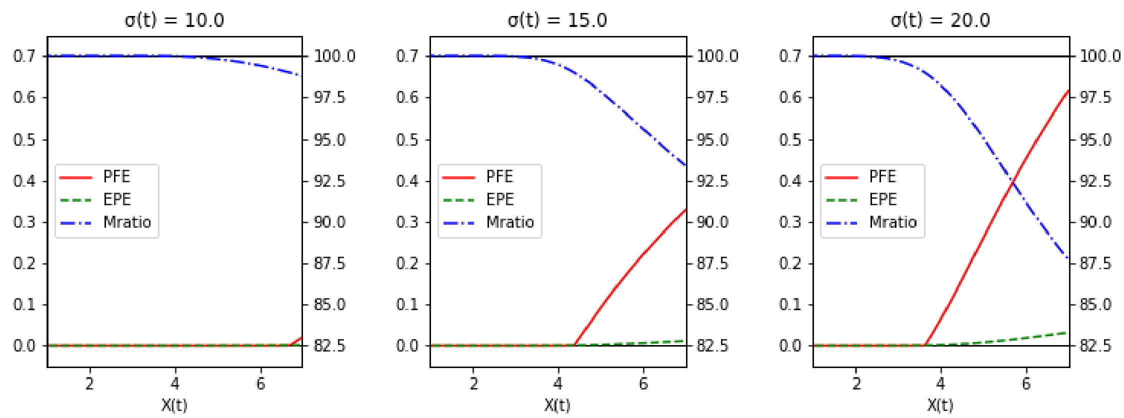

Figure 1: The counterparty risk indicators of PFE, EPE, and Mratio based on the ISDA SIMM method for the values of $X(t)$ ranging from 1% to 7%, for three values of the pseudo volatility 10% (left), 15% (middle), and 20% (right), respectively. The scale of PFE and EPE is on the left-hand side, while that of Mratio on the right-hand side (displayed in %).

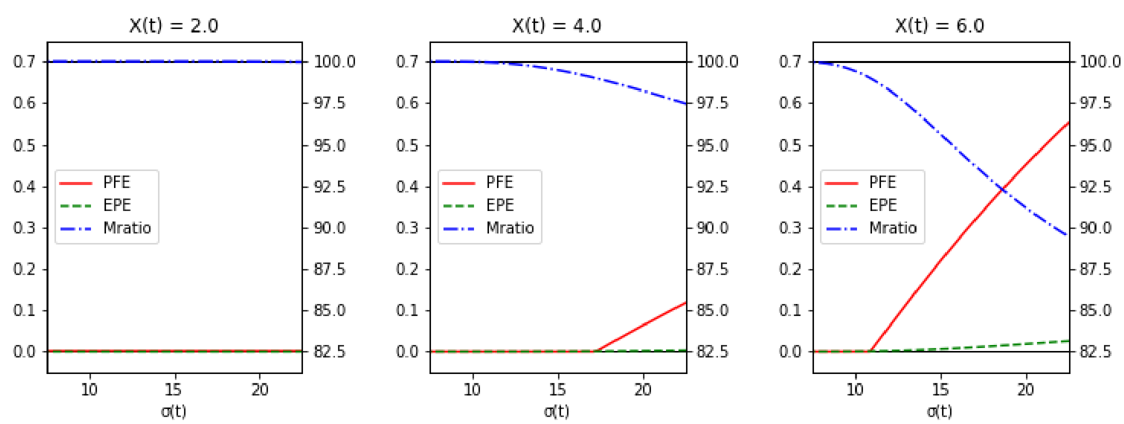

Figure 2: The counterparty risk indicators of PFE, EPE, and Mratio based on the ISDA SIMM method for the values of $\sigma(t)$ ranging from 5% to 25%, for three values of the underlying forward rate $X(t) = 2\%$ (left), 4% (middle), and 6% (right), respectively. The scale of PFE and EPE is on the left-hand side, while that of Mratio on the right-hand side (displayed in %).

can be approximately captured by decomposing it into three factors related to Delta, Vega, and Curvature as seen below.

$$
\begin{aligned}
&V(t+\delta_{\text{MPoR}}) - V(t) - \text{IM}^{\text{CSA}}(t) \\
&\approx \Big\{\partial_x V(t)\mu_X(t,X(t),\sigma(t))\delta_{\text{MPoR}} + A_X(t)Z_X - \text{IM}^{\text{Delta}}(t)\Big\} \\
&+ \{\partial_\sigma V(t)\mu_\sigma(t,X(t),\sigma(t))\delta_{\text{MPoR}} + A_\sigma(t)Z_\sigma - \text{IM}^{\text{Vega}}(t)\} \\
&+ \Big\{A_C(t) - [\partial_x V(t)\mu_X(t,X(t),\sigma(t)) + \partial_\sigma V(t)\mu_\sigma(t,X(t),\sigma(t))]\,\delta_{\text{MPoR}} + \Big(\text{IM}^{\text{Delta}}(t) + \text{IM}^{\text{Vega}}(t)\Big)\Big\}.
\end{aligned} \tag{23}
$$

Let

$$
\begin{aligned}
&L^{\text{Delta}}(t,\delta_{\text{MPoR}}) := \partial_x V(t)\mu_X(t,X(t),\sigma(t))\delta_{\text{MPoR}} + A_X(t)Z_X - \text{IM}^{\text{Delta}}(t), \\
&L^{\text{Vega}}(t,\delta_{\text{MPoR}}) := \partial_\sigma V(t)\mu_\sigma(t,X(t),\sigma(t))\delta_{\text{MPoR}} + A_\sigma(t)Z_\sigma - \text{IM}^{\text{Vega}}(t), \\
&L^{\text{Cvtr}}(t,\delta_{\text{MPoR}}) \\
&:= A_C(t) - [\partial_x V(t)\mu_X(t,X(t),\sigma(t)) + \partial_\sigma V(t)\mu_\sigma(t,X(t),\sigma(t))]\,\delta_{\text{MPoR}} + \Big(\text{IM}^{\text{Delta}}(t) + \text{IM}^{\text{Vega}}(t)\Big) \\
&= \Big\{\partial_t V(t) + \frac{1}{2}\partial^2_{xx}V(t)\sigma_X(t,X(t),\sigma(t))^2 + \frac{1}{2}\partial^2_{\sigma\sigma}V(t)\sigma_\sigma(t,X(t),\sigma(t))^2 \\
&+ \partial^2_{x\sigma}V(t)\rho\sigma_X(t,X(t),\sigma(t))\sigma_\sigma(t,X(t),\sigma(t))\Big\}\delta_{\text{MPoR}} - \text{IM}^{\text{Cvtr}}(t).
\end{aligned}
$$

We now like to discuss 99 % of the future exposure (FE), which can be negative, for each factor $L^*(t,\delta_{\text{MPoR}})$ in terms of the factorial decomposition of risk. Remark, however, that $L^{\text{Cvtr}}(t,\delta_{\text{MPoR}})$ does not include any random variable, so it is considered to contribute as a constant for fixed t.

For that purpose, we denote by $99\%\text{-FE}^{\text{Delta}}(t;\delta_{\text{MPoR}})$ (resp. $99\%\text{-FE}^{\text{Vega}}(t;\delta_{\text{MPoR}})$) the 99 % of the PFE for the factor related to Delta (resp. Vega) given as

$$
\begin{aligned}
99\%\text{-FE}^{\text{Delta}}(t;\delta_{\text{MPoR}}) &\approx \Big(\mu_X(t,X(t),\sigma(t))\delta_{\text{MPoR}} + 2.33\sigma(t)X(t)^\beta\sqrt{\delta_{\text{MPoR}}} - \text{RW}\Big)\,\partial_x V(t), \\
99\%\text{-FE}^{\text{Vega}}(t;\delta_{\text{MPoR}}) &\approx \Big(\mu_\sigma(t,X(t),\sigma(t))\delta_{\text{MPoR}} + 2.33\sigma(t)\nu\sqrt{\delta_{\text{MPoR}}} - \sigma(t)\text{VRW}\Big)\,\partial_\sigma V(t),
\end{aligned} \tag{24}
$$

However, we should remark

$$
\text{PFE}^{\text{CSA}}(t;\delta_{\text{MPoR}}) \neq 99\%\text{-FE}^{\text{Delta}}(t;\delta_{\text{MPoR}}) + 99\%\text{-FE}^{\text{Vega}}(t;\delta_{\text{MPoR}}) + L^{\text{Cvtr}}(t,\delta_{\text{MPoR}}).
$$

Under the same conditions as in subsection 4.1, we compute 99 % of FE of each factor for the values of $X(t)$ ranging from 1% to 7%, for three cases of the pseudo volatility: for the case of normal period with $\sigma(t)$ =10% or 15% and for the case of shock period with $\sigma(t)$ =20%, respectively. The result of factor decomposition is shown in Figure 3.

Also, we compute 99 % of FE of each factor for the values of $\sigma(t)$ ranging from 5% to 25% for the out-of-the money case with $X(t) = 2\%$ and for the in-the-money case with $X(t) = 4\%$ and 6%, respectively. The consequece is displyed in Figure 4.

Both the figures indicate that the Delta-related factor contributes most significantly to the increase in PFE when the pseudo volatility is relatively high.

In the actual interest rate market environment, a sudden rise in inflation may force the central bank to reduce quantitative easing (tapering) or raise interest rates sharply. Since the volatility and the forward rates are likely to increase simultaneously under such fluctuations in the financial environment, the rapid manifestation of counterparty risk, which is not preserved by the IM in practice, cannot be ruled out, as implied by our numerical analysis as is seen above.

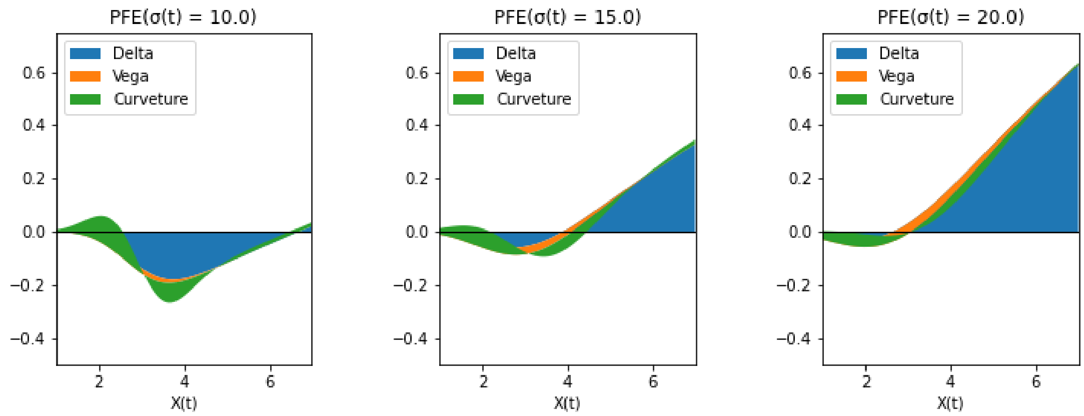

Figure 3: The 99 % of FE of each factor related to Delta, Vega, and Curvature for the values of $X(t)$ ranging from 1% to 7%, for three values of the pseudo volatility $\sigma(t)$ =10% (left), 15% (middle), and 20% (right), respectively.

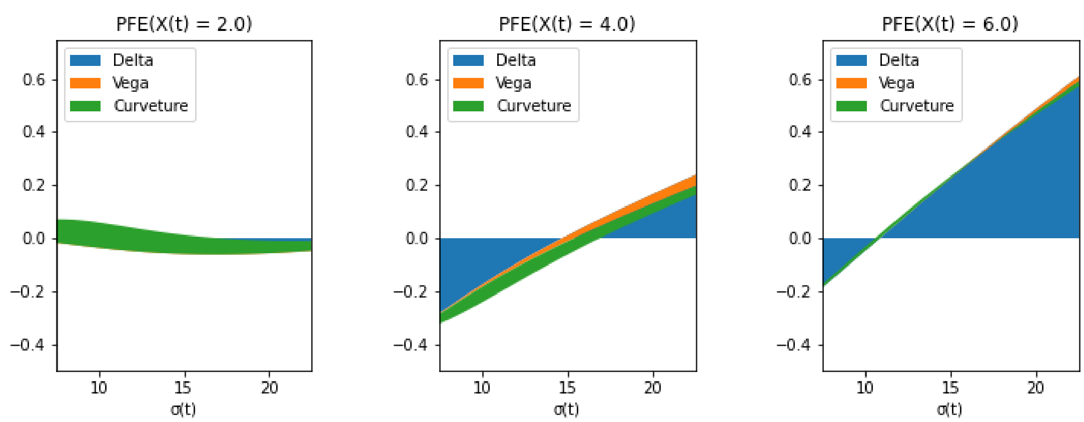

Figure 4: The 99 % of FE of each factor related to Delta, Vega, and Curvature for the values of $\sigma(t)$ ranging from 5% to 25% for the out-of-the money case with $X(t) = 2\%$ and for the in-the-money case with $X(t) = 4\%$ and 6%, respectively.

5 Another Application of Our model to Credit Valuation Adjustments

If the counterparty to a derivative transaction cannot be asked to provide margin or collateral, it is common to calculate the "Credit Valuation Adjustment (CVA)" as "*the difference between the value of the derivative transaction with and without considering the possibility of counterparty's default under the risk neutral probability measure,*" and to see the CVA as a counterparty risk indicator, as mentioned in Green [5] and so forth[7].

In this section, we do not go into the issue of CVA evaluation in earnest, but after reviewing the equation of CVA for the positive exposure $(V(t+\delta_{\mathrm{MPoR}}) - M(t))^+$ considering the MPoR δ_{MPoR} under some naive assumptions. Then, under the similar setting as in the last section we perform a simple analysis by numerically computing CVA for the payer swaptions under the SABR model.

5.1 CVA considering MPoR

In calculating CVA, we consider uncertainty about derivatives transactions and counterparty's default on the probability space $(\Omega, \mathcal{F}, \mathbf{Q})$, where the probability measure $\mathbf{Q}$ is a risk neutral probability measure. Here we consider a bilateral CVA under some simple assumptions and a close-out condition with the MPoR.

Next, we have to introduce the default-free interest rate, the risk-neutral hazard rate (default intensity) of yourself and the counterparty, and the recovery rate given default of the counterparty. Although it is preferable to model them by stochastic processes in order to obtain CVA precisely, we give them as constants for simplicity since we are not considering credit risk itself in this study. Note that implicit in this assumption is that the interest rate, the default risks of the parties, and the market value of the derivative are all independent.

Let r be the default-free interest rate, λ_B and λ_C be the risk-neutral hazard rate of yourself (B) and the counterparty (C) respectively, and R_C be recovery rate given default of the counterparty. We note that we regard the hazard rate as satisfying the followingλ_* $(* = B$ or $C)$ as satisfying $\mathbf{Q}(\tau_* > t) = e^{-\lambda_* t}$, where τ_B (resp. τ_C) is the default time of yourself (resp. the counterparty).

With a slight modification of the standard expression for bilateral CVA shown in subsection 3.3 of Green [5], we find that under the above conditions the expression for bilateral CVA, denoted by $\mathrm{CVA}(t;\delta_{\mathrm{MPoR}})$, at time t considering MPoR δ_{MPoR} is given by

$$\begin{aligned}\mathrm{CVA}(t;\delta_{\mathrm{MPoR}}) &= (1-R_C)\,\lambda_C \int_t^T \mathrm{DF}(t,u)\mathbf{E}^{\mathbf{Q}}\left[(V(u+\delta_{\mathrm{MPoR}}) - M(u))^+ \mid \mathcal{F}_t\right] du \qquad (25)\\ &= (1-R_C)\,\lambda_C \int_t^T \mathrm{DF}(t,u)\mathbf{E}^{\mathbf{Q}}\left[(V(u+\delta_{\mathrm{MPoR}}) - V(u) - \mathrm{IM}(u))^+ \mid \mathcal{F}_t\right] du,\end{aligned}$$

where $\mathrm{DF}(t,u)$ $(u \in [t,T])$ stands for the default risk adjusted discount factor given by

$$\mathrm{DF}(t,u) = e^{-r(u+\delta_{\mathrm{MPoR}}-t)-(\lambda_B+\lambda_C)(u-t)}.$$

Note that the close-out amount must be discounted at the default-free interest rate from the close-out time, while the counterparty's default are supposed to happen at time u.

[7] In addition to counterparty risk (CVA), costs or benefits associated with own credit risk (DVA), funding (FVA), margin (MVA), regulatory capital (KVA), and so forth, are now considered in valuation adjustments. Collectively, these are referred to as XVA, which is important for financial institutions to accurately calculate and manage because it is often recognized for accounting purposes. See for example Gregory citegregory2020xva for XVA.

As you see, the conditional expectation term in the integrand of (25) can be transformed into

$$\begin{aligned}
&\mathbf{E}^{\mathbf{Q}}\left[(V(u+\delta_{\mathrm{MPoR}})-M(u))^{+}\mid\mathcal{F}_t\right]\\
&=\mathbf{E}^{\mathbf{Q}}\left[\mathbf{E}^{\mathbf{Q}}\left[(V(u+\delta_{\mathrm{MPoR}})-M(u))^{+}\mid\mathcal{F}_u\right]\mid X(t),\sigma(t)\right]\\
&=\mathbf{E}^{\mathbf{Q}}\left[\mathrm{EPE}^{\mathrm{CSA},\mathbf{Q}}\left(u;\delta_{\mathrm{MPoR}}\right)\mid X(t),\sigma(t)\right], \qquad (26)
\end{aligned}$$

where we remark that the EPE given in (4) is caluculated here under the risk-neutral measure $\mathbf{Q}$ instead of $\mathbf{P}$.

Hence we can approximately calculate this conditional expectation term for the example of payer swaption in the SABR model by using (14) and (20) - (22) with $\theta_X(t)=\theta_\sigma(t)=0$.

5.2 Numerical analysis for CVA

The parameters for numerical calculation of CVA are assumed to be $r=0.03, \lambda_B=0.005, \lambda_C=0.01$, and $R_C=0.4$. The integral part is discretized by the time step $\Delta=1/365$, so we calculate the CVA at time t with the time to maturity $T-t=1$ as follows.

Our numerical calculations for CVA are performed using the following algorithm based on the Monte Carlo simulation with 1,000 trials. Since the objective is not to obtain the exact value of CVA, but to roughly see the trend of CVA for different values of $X(t)$ and $\sigma(t)$, the number of trials is limited to only 1,000 for computational efficiency.

1. Give the initial values $X(t)$ and $\sigma(t)$ at initial time t.

2. For n-th trial ($n=1,2,\ldots,1000$), we set $X^{(n)}(t)=X(t), \sigma^{(n)}(t)=\sigma(t)$ and then generate pseudo-random numbers for correlated normal variables Z_X and Z_σ and then inductively obtain discretized paths of the underlying forward rate $\{X^{(n)}(t+k\Delta)\}$ and the pseudo volatility $\{\sigma^{(n)}(t+k\Delta)\}$ ($k=1,2,\ldots,365$) with the following formulas

$$\begin{aligned}
X^{(n)}(t+k\Delta)&=X^{(n)}(t+(k-1)\Delta)+\sigma^{(n)}(t+(k-1)\Delta)\left(X^{(n)}(t+(k-1)\Delta)\right)^{\beta}\sqrt{\Delta}z_x^{(n,k)},\\
\sigma^{(n)}(t+k\Delta)&=\sigma^{(n)}(t+(k-1)\Delta)+\nu\sigma^{(n)}(t+(k-1)\Delta)\sqrt{\Delta}z_\sigma^{(n,k)},
\end{aligned}$$

   where $\{(z_X^{(n,k)}, z_\sigma^{(n,k)})\}_{k=1,2,\ldots,365}$ are a series of pairs of generated pseudo-random numbers of correlated normal variables (Z_X, Z_σ).

3. Calculate n-the samples of the conditional expectation in (26) given by

$$\mathrm{EPE}^{(n),\mathrm{CSA},\mathbf{Q}}\left(t+k\Delta;\delta_{\mathrm{MPoR}}\right)\ \ (k=1,2,\ldots,365),$$

   by substituting, at each time $t+k\Delta$, the randomly generated values $X^{(n)}(t+k\Delta)$ and $\sigma^{(n)}(t+k\Delta)$ into the approximate EPE formula (14) with the components (20) - (22) obtained in the SABR model.

4. Finally achieve the CVA approximately from the following formula.

$$\begin{aligned}
\mathrm{CVA}(t;\delta_{\mathrm{MPoR}})\approx(1-0.40)\times 0.01\times\frac{1}{365}\sum_{k=1}^{365}e^{-0.03\times\frac{k+14}{365}-(0.005+0.01)\times\frac{k}{365}}\\
\times\frac{1}{1000}\sum_{n=1}^{1000}\mathrm{EPE}^{(n),\mathrm{CSA},\mathbf{Q}}\left(t+\frac{k}{365};\delta_{\mathrm{MPoR}}\right).
\end{aligned}$$

The results are shown in Table 3. It can be seen that the CVA is likely to increase as the underlying forward rate $X(t)$ or the pseudo volatility $\sigma(t)$ increases. In addition, It is suggested that the magnitude of the increase depends on the level of ν. This seems because the parameter ν affects $X(t)$ and $\sigma(t)$ over time, and thus the exposure throughout the option period.

Table 3: The CVA calculated by the simulation. The CVA is calculated using the algorithm described above for several pairs of the inintial value $(X(t), \sigma(t))$ for values 10% and 30% of the parameter ν, respectively. The values of CVA are displayed in 10^{-6} %.

ν	$\sigma(t)$	$X(t) = 2\%$	$X(t) = 4\%$	$X(t) = 6\%$
10%	10%	0.12	0.91	9.66
	15%	0.67	21.38	103.46
	20%	6.08	89.08	308.01
30%	10%	0.25	5.01	22.52
	15%	3.32	41.10	140.81
	20%	17.99	131.26	385.35

6 Concluding Remarks

Regulations for OTC derivatives trading, which were initiated after the G20 Pittsburgh Summit (2009), triggered by the financial crisis, have been completed with the full enforcement of the OTC derivatives trade reporting system (2010), central clearing obligation (2012), electronic trading platform usage obligation (2015), variation margin requirements (2017) and initial margin requirements (2022), and as their consequences it is said that counterparty risks among major financial institutions have been significantly reduced. Especially the impact of defaults of individual financial institution on the financial system has been reduced because the majority of derivatives transactions are underwritten by CCPs through central clearing. Also, counterparty risks for OTC derivatives transactions that are not underwritten by CCPs have been reduced since the variation margin (VM) as well as the initial margin (IM) have been mandated by the regulations.

However, the IM, which protects the positive exposure of derivative transactions during the margin period of risk (MPoR), can only be transferred until just before the counterparty's default. Therefore, the regulation requires that 99% of PE fluctuation be preserved. On the other hand, in practice, the IM requirement has been calculated using a simple calculation method based on the ISDA SIMM. As a result, there may be a discrepancy between the regulation and practice in some cases, and concerns have been raised that the practical SIMM-based IM may not be up to the level required by the regulation.

In this study, we first give a framework to quantitatively evaluate the discrepancy between the 99% of PE required by the regulation and the simplified method based on the ISDA SIMM used in practice with respect to the calculation method of IM considering MPoR. We then, under a general stochastic volatility model, derive the approximate formula for PFE and EPE, which are common indicators of counterparty risk as well as Mratio, which is newly introduced as a conservation ratio of IM to the variation of exposure of derivatives transactions during the MPoR.

Moreover, we execute some numerical analyses of the counterparty risk indicators for payer swaptions (European call options) under the SABR model, which is a type of stochastic volatility model. It is suggested that if the position is in-the-money and the volatility increases, the SIMM based IM may not be sufficient to avoid counterparty risk, while it performs well under normal conditions with low volatility.

In addition, we attempt to decompose the PFE calculated by ISDA SIMM into three factors related to Delta, Vega, and Curvature. We also show that the framework we prepared can be applied to CVA calculation.

In the real market environment, interest rates and the volatility can rise simultaneously due to monetary policy of sharply raising interest rates, for example. In such cases, the counterparty risk can become apparent. Even if the counterparty does not default, the CVA may result in accounting losses. In such a sense, this study reaffirms the importance of counterparty risk management even in fully margined derivatives transactions.

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A Derivation of Greeks for the price of European call option price under the SABR model

Here we discuss how to derive the above Greeks for the pricing formula (19) of European call option under the SABR model so as to calculate the counterparty risk indicators like PFE, EPE, and Mratio.

First we notice

$$\frac{\partial}{\partial y}\phi\left(d(t)\right) = \frac{\partial d}{\partial y}\frac{\partial}{\partial d}\left(\frac{1}{\sqrt{2\pi}}e^{-\frac{d(t)^2}{2}}\right) = -d(t)\frac{1}{\sqrt{2\pi}}e^{-\frac{d(t)^2}{2}}\frac{\partial d}{\partial y}(t) = -d(t)\phi\left(d(t)\right)\frac{\partial d}{\partial y}(t)$$
$$= -\frac{X(t)-K}{\sigma_N(t)\sqrt{T-t}}\phi\left(d(t)\right)\frac{\partial d}{\partial y}(t).$$

Then the first-order partial derivatives, “Delta” $\left(\frac{\partial V}{\partial x}\right)$, “Vega” $\left(\frac{\partial V}{\partial \sigma}\right)$ and “Theta” $\left(\frac{\partial V}{\partial t}\right)$, can be obtained as follows.

$$\partial_x V(t) = \Phi\left(d(t)\right) + (X(t)-K)\frac{\partial}{\partial x}\Phi(d) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial x}(t) + \sigma_N(t)\sqrt{T-t}\frac{\partial}{\partial x}\phi\left(d(t)\right)$$
$$= \Phi\left(d(t)\right) + (X(t)-K)\phi\left(d(t)\right)\frac{\partial d}{\partial x}(t) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial x}(t) - (X(t)-K)\phi\left(d(t)\right)\frac{\partial d}{\partial x}(t)$$
$$= \Phi\left(d(t)\right) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial x}(t),$$

$$\partial_\sigma V(t) = (X(t)-K)\frac{\partial}{\partial \sigma}\Phi\left(d(t)\right) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial \sigma}(t) + \sigma_N(t)\sqrt{T-t}\frac{\partial}{\partial \sigma}(t)\phi\left(d(t)\right)$$
$$= (X(t)-K)\phi\left(d(t)\right)\frac{\partial d}{\partial \sigma}(t) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial \sigma}(t) - (X(t)-K)\phi\left(d(t)\right)\frac{\partial d}{\partial \sigma}(t)$$
$$= \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial \sigma}(t),$$

$$\partial_t V(t) = (X(t)-K)\frac{\partial}{\partial t}\Phi\left(d(t)\right) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial t}(t) - \frac{\sigma_N(t)\phi\left(d(t)\right)}{2\sqrt{T-t}} + \sigma_N(t)\sqrt{T-t}\frac{\partial}{\partial t}\phi\left(d(t)\right)$$
$$= (X(t)-K)\phi\left(d(t)\right)\frac{\partial d}{\partial t}(t) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial t}(t) - \frac{\sigma_N(t)\phi\left(d(t)\right)}{2\sqrt{T-t}}$$
$$- (X(t)-K)\phi\left(d(t)\right)\frac{\partial d}{\partial t}(t)$$
$$= \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial t}(t) - \frac{\sigma_N(t)\phi\left(d(t)\right)}{2\sqrt{T-t}}.$$

Before discussing the second-order partial derivatives, we remark that

$$\frac{\partial d}{\partial x}(t) = \frac{1}{\sigma_N(t)\sqrt{T-t}} + \frac{X(t)-K}{\sqrt{T-t}}\frac{\partial \sigma_N}{\partial x}(t)\frac{\partial}{\partial \sigma_N}\sigma_N(t)^{-1} = \frac{1}{\sigma_N(t)\sqrt{T-t}} - \frac{X(t)-K}{\sigma_N(t)^2\sqrt{T-t}}\frac{\partial \sigma_N}{\partial x}(t)$$
$$= \frac{1}{\sigma_N(t)\sqrt{T-t}}\left(1 - \frac{X(t)-K}{\sigma_N(t)}\frac{\partial \sigma_N}{\partial x}(t)\right),$$

$$\frac{\partial d}{\partial \sigma}(t) = \frac{X(t)-K}{\sqrt{T-t}}\frac{\partial \sigma_N}{\partial \sigma}(t)\frac{\partial}{\partial \sigma_N}\sigma_N(t)^{-1} = -\frac{X(t)-K}{\sigma_N(t)^2\sqrt{T-t}}\frac{\partial \sigma_N}{\partial \sigma}(t).$$

Thus the second-order partial derivatives “Gamma” $\left(\frac{\partial^2 V}{\partial x^2}\right)$, “Volga” $\left(\frac{\partial^2 V}{\partial x \partial \sigma}\right)$, and “Vanna”

$\left(\dfrac{\partial^2 V}{\partial \sigma^2}\right)$, are also obtained as follows.

$$
\begin{aligned}
\partial^2_{xx} V(t) &= \frac{\partial}{\partial x}\partial_x V(t) = \frac{\partial}{\partial x}\left(\Phi\left(d(t)\right) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial x}(t)\right) \\
&= \phi\left(d(t)\right)\frac{\partial d}{\partial x}(t) - \sqrt{T-t}\frac{\partial \sigma_N}{\partial x}(t)\frac{\partial}{\partial x}\phi\left(d(t)\right) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial^2 \sigma_N}{\partial x^2}(t) \\
&= \phi\left(d(t)\right)\frac{\partial d}{\partial x}(t) - \frac{X(t)-K}{\sigma_N(t)}\phi\left(d(t)\right)\frac{\partial d}{\partial x}(t)\frac{\partial \sigma_N}{\partial x}(t) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial^2 \sigma_N}{\partial x^2}(t) \\
&= \left(1 - \frac{X(t)-K}{\sigma_N(t)}\frac{\partial \sigma_N}{\partial x}(t)\right)\frac{\phi\left(d(t)\right)}{\sigma_N(t)\sqrt{T-t}}\left(1 - \frac{X(t)-K}{\sigma_N(t)}\frac{\partial \sigma_N}{\partial x}(t)\right) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial^2 \sigma_N}{\partial x^2}(t) \\
&= \left(1 - \frac{X(t)-K}{\sigma_N(t)}\frac{\partial \sigma_N}{\partial x}(t)\right)^2\frac{\phi\left(d(t)\right)}{\sigma_N(t)\sqrt{T-t}} + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial^2 \sigma_N}{\partial x^2}(t),
\end{aligned}
$$

$$
\begin{aligned}
\partial^2_{x\sigma} V(t) &= \frac{\partial}{\partial \sigma}\partial_x V(t) = \frac{\partial}{\partial \sigma}\left(\Phi\left(d(t)\right) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial x}(t)\right) \\
&= \phi\left(d(t)\right)\frac{\partial d}{\partial \sigma}(t) - \sqrt{T-t}\frac{\partial \sigma_N}{\partial x}(t)\frac{\partial}{\partial \sigma}\phi\left(d(t)\right) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial^2 \sigma_N}{\partial x \partial \sigma}(t) \\
&= \phi\left(d(t)\right)\frac{\partial d}{\partial \sigma}(t) - \frac{X(t)-K}{\sigma_N(t)}\phi\left(d(t)\right)\frac{\partial d}{\partial \sigma}(t)\frac{\partial \sigma_N}{\partial x}(t) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial^2 \sigma_N}{\partial x \partial \sigma}(t) \\
&= -\left(1 - \frac{X(t)-K}{\sigma_N(t)}\frac{\partial \sigma_N}{\partial x}(t)\right)\phi\left(d(t)\right)\frac{X(t)-K}{\sigma_N(t)^2\sqrt{T-t}}\frac{\partial \sigma_N}{\partial \sigma}(t) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial^2 \sigma_N}{\partial x \partial \sigma}(t),
\end{aligned}
$$

$$
\begin{aligned}
\partial^2_{\sigma\sigma} V(t) &= \frac{\partial}{\partial \sigma}\partial_\sigma V(t) = \frac{\partial}{\partial \sigma}\left(\sqrt{T-t}\phi\left(d(t)\right)\frac{\partial \sigma_N}{\partial \sigma}(t)\right) \\
&= \sqrt{T-t}\frac{\partial \sigma_N}{\partial \sigma}(t)\frac{\partial}{\partial \sigma}\phi\left(d(t)\right) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial^2 \sigma_N}{\partial \sigma^2}(t) \\
&= -\frac{X(t)-K}{\sigma_N(t)}\phi\left(d(t)\right)\frac{\partial d}{\partial \sigma}(t)\frac{\partial \sigma_N}{\partial \sigma}(t) + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial^2 \sigma_N}{\partial \sigma^2}(t) \\
&= \frac{(X(t)-K)^2}{\sigma_N(t)^3\sqrt{T-t}}\phi\left(d(t)\right)\left(\frac{\partial \sigma_N}{\partial \sigma}(t)\right)^2 + \sqrt{T-t}\phi\left(d(t)\right)\frac{\partial^2 \sigma_N}{\partial \sigma^2}(t).
\end{aligned}
$$