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Dynamics of Market Spread of First-to-Default Swap in an Information-Based Approach

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Abstract

The market for complex credit derivatives such as First-to-default swaps (FtDS) has become less actively traded since the 2008 financial crisis, even though FtDS might still be used in certain bespoke transactions by specific financial institutions. However, FtDS, thanks to its structural simplicity, plays a crucial role in theoretical research as an important starting point for understanding other more complex multi-name credit derivatives. We investigate the dynamics of the market spread of the FtDS using an information-based credit risk model. Specifically, we derive stochastic differential equations satisfied by the market FtDS spread and the first-to-default hazard rate process within the information-based model, allowing us to consider how they are driven by the information flow. Our findings show that the dynamics of the market FtDS spread are explained by the quality of information on survival conditions, measured by the discrepancy between conditional expectations of each credit factor under different conditions related to the first default time.

Keywords Dependent default, Information-based approach, First-to-default swap, Market spread

Mathematics Subject Classification (2020) 60H10, 60H30, 91B28

1 Introduction

We have been interested in how the price dynamics of defaultable securities, such as defaultable zero-coupon bonds and credit derivatives, are influenced by considering default risk contagion. In this study, we focus on the first-to-default swap (FtDS), a type of basket credit derivative, and investigate the dynamics of the market spread of FtDS, that is, the value of the credit default swap (CDS) spread that makes the protection leg and the premium leg equivalent at any time during the contract period.

In existing studies, numerous findings exist regarding the pricing of the FtDS; to mention a few: Frey and Backhaus [10], Liang, Ma, Wang et al. [13], Schönbucher [16], and Zheng and Jiang [19]. Most of these results deal only with static cases, focusing solely on computing prices

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at a particular time point without considering temporal evolution. Few results offer a dynamic perspective, particularly concerning stochastic differential equations governing the price process. In the case of a single-name CDS, Bielecki, Jeanblanc, and Rutkowski [2] derive the dynamics of the market CDS spread without any specific model, just in terms of a semimartingale decomposition. The result on the single-name market CDS spread can be extended to the case of the market FtDS spread, allowing for a reinterpretation of the dynamics of the market spread of FtDS. However, we aim to further investigate the dynamics of the market FtDS spread more specifically via the stochastic differential equations satisfied by the market FtDS spread. To do so, we need to assume a specific credit risk model. For this purpose, we use a multi-name credit risk model with an information-based approach as proposed by Nakagawa and Takada [15].

The information-based approach, initially introduced by Brody et al. [4], proposes a method for representing the dynamics of asset prices by modeling the flow of market information. In their framework, it is assumed that market information comprises the future cash flow values associated with the given assets. However, market participants are unable to observe this information directly and must contend with the presence of noise. Based on the available information related to the anticipated payouts of the given asset, market participants make their best efforts to estimate the value of the upcoming cash flows. These estimates, in turn, influence decisions regarding transactions, subsequently triggering movements in the asset's price. The information-based approach can also be applied to depict the perceived probability of default, which may vary based on the information flow representing market sentiments regarding default risk. The single-name case has been thoroughly studied by Brody et al. [5], and extended to multi-name cases by Nakagawa and Takada [15] to derive the dynamics of a pair of defaultable zero-coupon bonds in an explicit form.

The primary reason for focusing on the information-based approach to model the market spread of FtDS is its ability to easily incorporate the structure of default contagion. Default contagion is the phenomenon which is often observed in the market. One default event can negatively impact the credit quality of other active companies, potentially leading to further defaults in the worst case. From the viewpoint of credit risk management, numerous researchers and practitioners have been interested in modeling for the analysis of default contagion (see, for example, [10], [11], [12], [17], and [19]). This interest is primarily motivated by the belief that accurate estimations of default contagion play a pivotal role in refining the measurement of counterparty risk, valuation procedures, and hedging strategies for credit-risky assets dependent on multiple entities. As Duffie [8] suggests, if the default intensity processes of all the reference companies are independent, the market spread of FtDS is merely dependent on the aggregate of default intensities within the reference pool. In general, however, default dependencies must be taken into account, and therefore the dynamics of the market FtDS spread are considered to be related to these default dependencies. Our research focuses precisely on this aspect: how does default contagion influence the stochastic dynamics of the market spread of the FtDS? In our information-based credit risk model, it is possible to incorporate the potential for default contagion among multiple obligors by introducing correlations between latent variables called "credit-related individual factor", which play a role in determining the default times of each obligor.

In this study, we first discuss the dynamics of the market FtDS spread in a model-free manner, assuming a stylized FtDS instead of an actively traded one. We then introduce the information-based multi-name credit risk model developed by Nakagawa and Takada [15]. Using this model, we derive the stochastic differential equations (SDEs) satisfied by the market FtDS spread and the first-to-default hazard rate process to specifically examine their dynamics. As a result, the diffusion terms of the market FtDS spread and the first-to-default hazard rate process exhibit very similar structures. These diffusion terms depend on the discrepancy between the conditional expectations of each credit factor due to differences in conditions regarding the first default time.

The remainder of this paper is organized as follows: In Section 2, we establish a general mathematical model to describe a stylized FtDS market and review the valuation formula for the FtDS. Additionally, we derive a model-free stochastic differential equation (semimartingale decomposition) satisfied by the market FtDS spread. In Section 3, we introduce our information-based multi-name default risk model and prepare several mathematical tools from Nakagawa and Takada [15]. In Section 4, we introduce several additional symbols to succinctly express the results and use them to derive the stochastic differential equations satisfied by the market FtDS spread and the first-to-default hazard rate process. We also explore how the dynamics of the market FtDS spread and the first-to-default hazard rate process are influenced by the information flow. Section 5 and the Appendix provide the proofs of the propositions concerning the SDEs followed by the market FtDS spread and the first-to-default hazard rate, respectively.

2 First-to-Default Swap

A first-to-default swap (FtDS) is a type of basket credit derivative backed by a reference portfolio of credit-sensitive securities to protect against losses resulting from the first default in the reference portfolio. In this sense, the FtDS is a multi-name version of credit default swap (CDS). Specifically, the seller of the protection must compensate the buyer for its losses in the event of the first default in the reference portfolio before the maturity of the FtDS. Meanwhile, the buyer must periodically pay the seller a predetermined premium, called the FtDS spread, until maturity or the first default occurs.

In general, the FtDS is mathematically formulated as follows. Under the assumption of no arbitrage, we model a financial market that includes several defaultable instruments on a probability space $(\Omega, \mathcal{G}, \mathbb{P})$, which is rich enough to support some Brownian motions. We assume that $\mathbb{P}$ is a risk-neutral pricing measure. The pricing measure $\mathbb{P}$ cannot be uniquely specified only by the assumption of no arbitrage due to market incompleteness. In practice, however, this assumption is sufficient for the discussion that follows since one can imply some model parameters under the pricing measure by calibrating the obtained pricing model to the market data of corporate bonds or credit default swaps. In what follows, we suppose that all expectations are taken under the risk-neutral pricing measure $\mathbb{P}$.

Similar to Brody et al. [5], throughout the paper we assume that the credit risk-free interest rate process r_t is deterministic. Hence the credit risk-free saving account, denoted by $B_t := \exp(\int_0^t r_u du)$, is also deterministic. We note that for $t < s$,

$$B_s^{-1} B_t = \exp\left(-\int_t^s r_u du \right)$$

stands for the (deterministic) credit risk-free discount factor from time s to time t[1].

[1] It is possible to make the credit risk-free interest rate stochastic without affecting our discussion on credit risk

Let $n \in \mathbb{N}$. We suppose the FtDS whose reference portfolio consists of n obligors' debts and that the maturity time of the FtDS is fixed at $T \in (0, \infty)$. We specify the n obligors by their default times, or some nonnegative $\mathcal{G}$-random variables denoted by $\tau_1, \cdots, \tau_n$ respectively. In addition, we denote the first default time among them by $\tau_{(1)} := \min\{\tau_i \mid i \in [n]\} = \tau_1 \wedge \cdots \wedge \tau_n$. For notational convenience, we denote by $[n] := \{1, 2, \cdots, n\}$ a set of all the obligors in our universe.

Here we introduce some filtrations. Let $\{\mathcal{H}_t^i\}$ be the filtration generated by the default indicator process $\mathbf{1}_{\{\tau_i \le t\}}$ of each obligor $i \in [n]$, where $\mathcal{H}_t^i := \sigma(\tau_i \wedge s : 0 \le s \le t)$. Also, we denote by $\{\mathcal{H}_t\}$ the filtration of the whole default information given by $\mathcal{H}_t := \bigvee_{i=1}^n \mathcal{H}_t^i$. Moreover, let $\{\mathcal{F}_t\}$ be a filtration for the whole market information except for the occurrence of defaults. The filtration $\{\mathcal{F}_t\}$, which we hereafter call "the default-free market filtration," is formally defined here, and we will specify it using an information-based approach in the next section.

Since we have $\{\tau_{(1)} > t\} = \{\tau_1 > t, \cdots, \tau_n > t\}$, the conditional survival probability of the first default time $\tau_{(1)}$ given $\mathcal{F}_t$ defined by $\bar{F}_{(1)}(u;t) := \mathbb{P}\left(\tau_{(1)} > u \mid \mathcal{F}_t\right)$ can be viewed as

$$\mathbb{P}\left(\tau_{(1)} > u \mid \mathcal{F}_t\right) = \mathbb{P}\left(\tau_1 > u, \cdots, \tau_n > u \mid \mathcal{F}_t\right).$$

Let $\bar{F}(u_1, \ldots, u_n; t) := \mathbb{P}\left(\tau_1 > u_1, \cdots, \tau_n > u_n \mid \mathcal{F}_t\right)$ for $u_1, \cdots, u_n,\ t \ge 0$ be the conditional joint survival probability function. Thus we have $\bar{F}_{(1)}(u;t) = \bar{F}(u, \ldots, u; t)$. Also, we denote the conditional density of $\tau_{(1)}$ given $\mathcal{F}_t$ by $f_{(1)}(u;t) := -\frac{d}{du}\bar{F}_{(1)}(u;t)$. We remark that if the conditional joint survival probability is differentiable, the conditional density $f_{(1)}(u;t)$ can be represented by

$$f_{(1)}(u;t) = -\sum_{i=1}^{n} \partial_i \bar{F}\left(u, \cdots, u; t\right),$$

where ∂_i represents the partial derivative of the function $\bar{F}$ with respect to the i-th variable. (See for example Bielecki et al. [3] pp.164-165.) We suppose that the conditional density $f_{(1)}(u;t)$ exists hereafter in this section.

We also remark that $\bar{F}_{(1)}(t;t)$ is viewed as a bounded supermartingale, so the relation

$$\bar{F}_{(1)}(t;t) = 1 - \int_0^t f_{(1)}(u;t)du$$

is obtained as the unique Doob-Meyer decomposition of $\bar{F}_{(1)}(t;t)$. Thus according to subsection 3.8 in Bielecki et al. [3], the first-to-default hazard rate process $\{\lambda_t^{(1)}\}$ is defined by

$$\lambda_t^{(1)} := \frac{f_{(1)}(t;t)}{\bar{F}_{(1)}(t;t)}. \tag{1}$$

We also see $\lambda_t^{(1)} = \sum_{i=1}^{n} \tilde{\lambda}_t^i$, where $\tilde{\lambda}_t^i$ is the first-to-default intensity for obligor $i \in [n]$ given by

$$\tilde{\lambda}_t^i := \lim_{h \downarrow 0} \frac{1}{h} \frac{\mathbb{P}\left(t < \tau_i \le t+h, \tau_{(1)} > t \mid \mathcal{F}_t\right)}{\bar{F}_{(1)}(t;t)}.$$

modeling by introducing another information process as Section 2.2.2 of [18]. However, our main concern is modeling the first default, so we need to pay little attention to the risk-free rate dynamics.

We will now introduce the pricing model for the FtDS and specify what will be called the "market FtDS spread," which is the focus of our analysis. Let us mention in advance that we will formulate it in an abstract, stylized manner with continuous premium payments instead of discrete payments made in the actual transaction. Such a formulation is called "stylised" in subsection 4.3 of Capiński and Zastawniak [6]. It has the advantage of being mathematically more manageable, such as not taking into account the accrued interest.

The evaluation of FtDS is usually considered separately on the "Protection Leg" and "Premium Leg," similar to that of a single-name CDS. In recent CDS conventions, the premium payments are customarily set at a fixed annual spread of 1% or 5%, depending on the credit risk of the reference bond, and the discrepancy in the initial value of both legs is paid as an upfront premium. However, we will consider the spread that makes the initial value of both legs equal according to the old convention, without any upfront payment at the time of the contract.

Protection Leg

The protection seller has to compensate the protection buyer for the loss caused by the first default event in the reference portfolio if it occurs before the maturity T. We suppose that the compensation at the first default time is given by a constant θ. Then the present value at time $t \in [0,T]$ of the protection leg is given by

$$\mathbb{E}\left[B_{\tau_{(1)}}^{-1} B_t \theta \mathbf{1}_{\{t<\tau_{(1)}\leq T\}} \mid \mathcal{F}_t \vee \mathcal{H}_t\right].$$

Using some elementary credit risk calculation methods, we have

$$\begin{aligned}
&\mathbb{E}\left[B_{\tau_{(1)}}^{-1} B_t \theta \mathbf{1}_{\{t<\tau_{(1)}\leq T\}} \mid \mathcal{F}_t \vee \mathcal{H}_t\right] \\
&= \mathbf{1}_{\{t<\tau_{(1)}\}} \frac{1}{\mathbb{P}\left(\tau_1 > t \mid \mathcal{F}_t\right)} \mathbb{E}\left[B_{\tau_{(1)}}^{-1} B_t \theta \mathbf{1}_{\{t<\tau_{(1)}\leq T\}} \mid \mathcal{F}_t\right] \\
&= \mathbf{1}_{\{t<\tau_{(1)}\}} \frac{\theta}{\bar{F}_{(1)}(t;t)} \int_t^T B_u^{-1} B_t f_{(1)}(u;t) du.
\end{aligned}$$

Premium Leg

We assume that the protection buyer continuously pays a predetermined premium called the FtDS spread to the protection seller until the first default occurs or the FtDS matures.

If the FtDS spread is given by a constant $\bar{\kappa}^{(1)}$, the present value at time t of the premium leg is given by

$$\mathbb{E}\left[\int_t^T B_u^{-1} B_t \bar{\kappa}^{(1)} \mathbf{1}_{\{u<\tau_{(1)}\}} du \mid \mathcal{F}_t \vee \mathcal{H}_t\right].$$

Using some elementary credit risk calculation methods, we have

$$\begin{aligned}
&\mathbb{E}\left[\int_t^T B_u^{-1} B_t \bar{\kappa}^{(1)} \mathbf{1}_{\{u<\tau_{(1)}\}} du \mid \mathcal{F}_t \vee \mathcal{H}_t\right] \\
&= \mathbf{1}_{\{t<\tau_{(1)}\}} \frac{\bar{\kappa}^{(1)}}{\bar{F}_{(1)}(t;t)} \int_t^T B_u^{-1} B_t \mathbb{P}\left(u < \tau_{(1)} \mid \mathcal{F}_t\right) du \\
&= \mathbf{1}_{\{t<\tau_{(1)}\}} \frac{\bar{\kappa}^{(1)}}{\bar{F}_{(1)}(t;t)} \int_t^T B_u^{-1} B_t \bar{F}_{(1)}(u;t) du.
\end{aligned}$$

Market FtDS Spread

Here we define **the T-maturity market FtDS spread** $\kappa_t^{(1)}$ at time $t \in [0,T]$ as the level of the T-maturity FtDS spread that makes the premium leg with the spread and the protection leg equivalent at time t. The term "market" FtDS spread follows the definition of the market CDS spread in subsection 3.7.5 of Bielecki et al. [3]. We should remark the T-maturity market FtDS spread $\kappa_t^{(1)}$is different from the spread $\bar{\kappa}^{(1)}$ that is actually paid. Namely,

$$\kappa_t^{(1)} := \mathbf{1}_{\{t<\tau_{(1)}\}} \theta \frac{\int_t^T B_u^{-1} f_{(1)}(u;t) du}{\int_t^T B_u^{-1} \bar{F}_{(1)}(u;t) du}. \tag{2}$$

We can rewrite the market FtDS spread $\kappa_t^{(1)}$ given in (2) as

$$\kappa_t^{(1)} = \mathbf{1}_{\{t<\tau_{(1)}\}} \frac{\theta(t,T)}{A(t,T)},$$

where

$$\begin{aligned}
A(t,T) &:= \int_t^T B_u^{-1} \bar{F}_{(1)}(u;t) du, \\
\theta(t,T) &:= \theta \int_t^T B_u^{-1} f_{(1)}(u;t) du.
\end{aligned}$$

We note that $A(t,T)$ can be represented by the tower property of conditional expectation as follows.

$$\begin{aligned}
A(t,T) &= \int_t^T B_u^{-1} \mathbb{P}\left(\tau_1 > u, \cdots, \tau_n > u \mid \mathcal{F}_t\right) du \\
&= \int_t^T B_u^{-1} \mathbb{E}\left[\mathbb{P}\left(\tau_1 > u, \cdots, \tau_n > u \mid \mathcal{F}_u\right) \mid \mathcal{F}_t\right] du \\
&= \mathbb{E}\left[\int_t^T B_u^{-1} \bar{F}_{(1)}(u;u) du \mid \mathcal{F}_t\right] \\
&= \mathbb{E}\left[\int_0^T B_u^{-1} \bar{F}_{(1)}(u;u) du \mid \mathcal{F}_t\right] - \int_0^t B_u^{-1} \bar{F}_{(1)}(u;u) du.
\end{aligned}$$

Then we can see that the process $\{m_t^A\}$ defined by

$$m_t^A := \mathbb{E}\left[\int_0^T B_u^{-1}\bar{F}_{(1)}(u;u)du \mid \mathcal{F}_t\right] \tag{3}$$

is an $\{\mathcal{F}_t\}$-martingale. It follows

$$\begin{aligned} m_t^A &= \int_0^T B_u^{-1}\mathbb{E}\left[\mathbb{E}\left[\mathbf{1}_{\{\tau_1>u,\cdots,\tau_n>u\}} \mid \mathcal{F}_u\right] \mid \mathcal{F}_t\right] du \\ &= \int_0^T B_u^{-1}\mathbb{E}\left[\mathbf{1}_{\{\tau_1>u,\cdots,\tau_n>u\}} \mid \mathcal{F}_{u\wedge t}\right] du \\ &= \int_0^T B_u^{-1}\bar{F}_{(1)}(u;u\wedge t)du. \end{aligned}$$

Similarly, $\theta(t,T)$ can be represented as

$$\begin{aligned} \theta(t,T) &= \theta\mathbb{E}\Big[\int_t^T B_u^{-1}\Big(-\sum_i \partial_i \bar{F}(u,\cdots,u;u)\Big)du \mid \mathcal{F}_t\Big] \\ &= \theta\Bigg\{\mathbb{E}\Big[\int_0^T B_u^{-1}f_{(1)}(u;u)du \mid \mathcal{F}_t\Big] - \int_0^t B_u^{-1}f_{(1)}(u;u)du\Bigg\}. \end{aligned}$$

The process $\{m_t^\theta\}$ defined by

$$m_t^\theta := \mathbb{E}\Big[\int_0^T B_u^{-1}f_{(1)}(u;u)du \mid \mathcal{F}_t\Big] \tag{4}$$

is an $\{\mathcal{F}_t\}$-martingale, and it follows

$$m_t^\theta = \int_0^T B_u^{-1}f_{(1)}(u;u\wedge t)du.$$

Hence we can express

$$dA(t,T) = dm_t^A - B_t^{-1}\bar{F}_{(1)}(t;t)dt, \tag{5}$$

$$d\theta(t,T) = \theta\left\{dm_t^\theta - B_t^{-1}\bar{F}_{(1)}(t;t)\lambda_t^{(1)}dt\right\}. \tag{6}$$

The second equality follows from the definition of the first-to-default hazard rate process $\lambda_t^{(1)}$ in (1).

To conclude this section, we derive a semimartingale decomposition satisfied by the market FtDS spread processes $\{\kappa_t^{(1)}\}$ given in (2). The decomposition is almost the same as that for the spread of a single-name credit default swap presented at the end of subsection 3.7.5 of Bielecki et al. [3].

Theorem 2.1. *The market FtDS spread process $\{\kappa_t^{(1)}\}$ with maturity T given in (2) satisfies the following stochastic differential equation (SDE).*

$$\begin{aligned} d\kappa_t^{(1)} = \mathbf{1}_{\{t<\tau_{(1)}\}}\frac{1}{A(t,T)}\Bigg\{&B_t^{-1}\bar{F}_{(1)}(t;t)\left(\kappa_t^{(1)} - \theta\lambda_t^{(1)}\right)dt + \theta dm_t^\theta - \kappa_t^{(1)}dm_t^A \\ &+ \frac{\kappa_t^{(1)}d\langle m^A, m^A\rangle_t - \theta d\langle m^A, m^\theta\rangle_t}{A(t,T)}\Bigg\} + \kappa_{t-}^{(1)}d\mathbf{1}_{\{t<\tau_{(1)}\}}, \end{aligned} \tag{7}$$

where $\lambda_t^{(1)}$ *is the first-to-default hazard rate process defined in* (1)*.*

Proof. We apply Ito's formula to obtain

$$
\begin{aligned}
d\kappa_t^{(1)} &= \mathbf{1}_{\{t<\tau_{(1)}\}} d\left(\frac{\theta(t,T)}{A(t,T)}\right) + \kappa_{t-}^{(1)} d\mathbf{1}_{\{t<\tau_{(1)}\}} \\
&= \mathbf{1}_{\{t<\tau_{(1)}\}} \frac{\theta(t,T)}{A(t,T)} \Bigg(\frac{d\theta(t,T)}{\theta(t,T)} - \frac{dA(t,T)}{A(t,T)} + \frac{d\langle A(\cdot,T), A(\cdot,T)\rangle_t}{(A(t,T))^2} \\
&\qquad - \frac{d\langle \theta(\cdot,T), A(\cdot,T)\rangle_t}{\theta(t,T)A(t,T)} \Bigg) + \kappa_{t-}^{(1)} d\mathbf{1}_{\{t<\tau_{(1)}\}}. \qquad (8)
\end{aligned}
$$

Substituting the equations (5) and (6) into (8), we obtain

$$
\begin{aligned}
d\kappa_t^{(1)} = \mathbf{1}_{\{t<\tau_{(1)}\}} \frac{1}{A(t,T)} \Bigg\{ & \theta dm_t^{\theta} - B_t^{-1} \bar{F}_{(1)}(t;t) \theta \lambda_t^{(1)} dt \\
& - \kappa_t^{(1)} \big(dm_t^A - B_t^{-1} \bar{F}_{(1)}(t;t) dt \big) \\
& + \kappa_t^{(1)} \frac{d\langle A(\cdot,T), A(\cdot,T)\rangle_t}{A(t,T)} - \frac{d\langle \theta(\cdot,T), A(\cdot,T)\rangle_t}{A(t,T)} \Bigg\} \\
+ \ \kappa_{t-}^{(1)} d\mathbf{1}_{\{t<\tau_{(1)}\}}. &
\end{aligned}
$$

Since the quadratic covariations satisfy

$$
\begin{aligned}
d\langle A(\cdot,T), A(\cdot,T)\rangle_t &= d\langle m_\cdot^A, m_\cdot^A\rangle_t, \\
d\langle \theta(\cdot,T), A(\cdot,T)\rangle_t &= \theta d\langle m_\cdot^\theta, m_\cdot^A\rangle_t,
\end{aligned}
$$

we can obtain the equation given in (8).

□

We remark that the market FtDS spread process $\kappa_t^{(1)}$ gets closer to the product of the compensation at default and the first-to-default hazard rate process $\theta\lambda_t^{(1)}$ as $t \nearrow T$. Indeed, heuristically, the following can be obtained in a model-free manner.

$$
\begin{aligned}
\kappa_{T-}^{(1)} = \lim_{t\nearrow T} \kappa_t^{(1)} &= \lim_{t\nearrow T} \mathbf{1}_{\{t<\tau_{(1)}\}} \frac{\theta \frac{1}{T-t} \int_t^T B_u^{-1} f_{(1)}(u;t) du}{\frac{1}{T-t} \int_t^T B_u^{-1} \bar{F}_{(1)}(u;t) du} \\
&= \mathbf{1}_{\{T<\tau_{(1)}\}} \theta \frac{f_{(1)}(T;T)}{\bar{F}_{(1)}(T;T)} = \mathbf{1}_{\{T<\tau_{(1)}\}} \theta \lambda_T^{(1)}.
\end{aligned}
$$

To calculate (7) further we need to know some specific form of the conditional joint survival probability $\bar{F}(u_1,\ldots,u_n;t)$ of the default times. In other words, we need to introduce a model that clarifies default dependencies among the default times $\tau_1,\cdots,\tau_n$. In the next section, we introduce an information-based model of default times in order to achieve more specific representation of the SDE satisfied by the T-maturity market FtDS spread $\kappa_t^{(1)}$.

3 Information-based model of default times

In this section, we introduce our information-based model of default times according to Nakagawa and Takada [15]. We suppose that the probability space and other settings are given in the previous section.

At first, the information-based model characterizes the default times $\{\tau_i\}_{i\in[n]}$ in terms of some integrable random variables $\{Z_i\}_{i\in[n]}$. Here we think of Z_i as representing some credit-related individual factor for each obligor i. We will often call it the i-th credit factor for simplicity. Specifically, we define the i-th obligor's default time by

$$\tau_i := h_i^{-1}(Z_i),$$

where h_i is a continuous deterministic invertible increasing function with $\lim_{s\to 0} h_i(s) = -\infty$, $\lim_{s\to\infty} h_i(s) = +\infty$.

Let $p_0\left((z_j)_{j\in[n]}\right) = p_0(z_1,\ldots,z_n)$ is the unconditional joint density of the credit factors $(Z_1,\ldots,Z_n)$. For example, we can regard the i-th credit factor Z_i as $Z_i = \rho_i\bar{Z} + \sqrt{1-\rho_i^2}\varepsilon_i$, where $\bar{Z}$ and ε_i are independent standard normal variables representing a common factor and idiosyncratic factor respectively and $\rho_i \in [-1,1]$ is a parameter. Then the credit factors $\{Z_i\}_{i\in[n]}$ are specified as some correlated standard normal random variables, so p_0 can be regarded as an n-dimensional centered correlated normal density function. Indeed, we will suppose that p_0 is such a correlated normal density function for a numerical illustration later. We also remark that such specification of default times is analogue to the idea of so called ASFR Model (asymptotic single factor risk model).

Also, from another point of view, the formulation may be classified into incomplete-information structural approach such as Çetin, Jarrow, Protter and Yildirim [7], Duffie and Lando [9], and Nakagawa [14] for single-name case, and Benzoni, Collin-Dufresne, Goldstein and Helwege [1] for multi-name case.

Next, we introduce the concept of market information flow, whereby we can explicitly describe the amount of available information associated with the credit factor. We assume that market participants can only access partial information with inseparable noise. Hence we can concretely compose the default-free market filtration $\{\mathcal{F}_t\}$, which stands for the information available to the market participants, as shown below.

For each $i = 1,\ldots,n$, let $\{\xi_t^i\}$ $(i \in [n])$ be an i-th market information process associated with the credit factor Z_i, which is formulated as

$$\xi_t^i := \sigma_i t Z_i + B_t^i, \quad (i \in [n])$$

where $\sigma_i > 0$ is termed "information flow rate" (see [5]), and $\{B_t^i : i \in [n]\}$ is a set of n mutually independent standard Brownian motions, called "market noise," that are independent of all the credit factors $\{Z_i\}_{i\in[n]}$.

Then, we specify the default-free market filtration $\{\mathcal{F}_t\}$ as the natural filtration generated from the observable information processes $\{\xi_t^i\}$, namely,

$$\mathcal{F}_t := \sigma(\xi_s^i : 0 \le s \le t, i \in [n]).$$

The default information filtrations $\{\mathcal{H}_t^i\}$ and $\mathcal{H}_t := \bigvee_{i=1}^n \mathcal{H}_t^i$ are the same as those given in the previous section. Finally, we define $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t$ for any $t \geq 0$ and view the filtration $\{\mathcal{G}_t\}$ as the total information available to the market participants. Now that the default-free market filtration $\{\mathcal{F}_t\}$ is specifically given, we can proceed with the calculation of the conditional joint survival probability $\bar{F}(u_1, \ldots, u_n; t)$ for the first default time. This is done in order to analyze the dynamics of the T-maturity market FtDS spread $\kappa_t^{(1)}$ presented at the end of the previous section.

As shown in [15], we remark that the information process $\{\xi_t^i\}$ has the Markov property with respect to the filtration $\{\mathcal{F}_t\}$ because we have

$$\mathbb{P}(\xi_t^i \leq x \mid \xi_s^i, \xi_{s_1}^i, \xi_{s_2}^i, \cdots, \xi_{s_k}^i) = \mathbb{P}(\xi_t^i \leq x \mid \xi_s^i),$$

for any collection of times $t, s, s_1, \ldots, s_k$ with $t \geq s \geq s_1 \geq s_2 \geq \cdots \geq s_k > 0$. Thus we can show the following proposition similar to the proof of Proposition 2.3 in [15]. Henceforth, we shall use the following notation.

$$\mathcal{E}_t(\xi^i; z_i) := \exp\left(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2\right).$$

Proposition 3.1. *On the set $\{\tau_1 > t, \cdots, \tau_n > t\}$, that is, if no default happens until t, the conditional joint survival probability of the first default time $\bar{F}(u_1, \ldots, u_n; t)$ is given as follows. For any $u_1, \ldots, u_n$ $(\geq t)$, we have*

$$\begin{aligned}\bar{F}(u_1, \ldots, u_n; t) &= \mathbb{P}\left(\tau_1 > u_1, \cdots, \tau_n > u_n \mid \ \{\xi_t^j\}_{j\in[n]}\right) \\ &= \frac{\displaystyle\int_{\mathbb{R}^n} \prod_{j=1}^n \mathbf{1}_{\{z_j > h_j(u_j)\}} p_0\left((z_j)_{j\in[n]}\right) \prod_{j=1}^n \mathcal{E}_t(\xi^j; z_j)(dz_j)_{j\in[n]}}{\displaystyle\int_{\mathbb{R}^n} p_0\left((z_j)_{j\in[n]}\right) \prod_{j=1}^n \mathcal{E}_t(\xi^j; z_j)(dz_j)_{j\in[n]}}.\end{aligned}$$

Thus the stochastic dynamics of the T-maturity market FtDS spread $\{\kappa_t^{(1)}\}$ given in (7) can be represented more explicitly in our information-based credit risk model through further calculations using Proposition 3.1. In the following sections, we aim to obtain a more specific representation than (8) of the stochastic differential equation satisfied by the T-maturity market FtDS spread $\kappa_t^{(1)}$ with our information-based model.

Figure 1 shows simultaneously simulated sample trajectories on the interval [0,0.5] of the FtDS spread process $\kappa_t^{(1)}$(dotted line) and the first to default intensity process $\lambda_t^{(1)}$(solid line) for the reference portfolio $\{1, 2\}$. We set $r_t \equiv 0.01$(constant), $\sigma_1 = 0.1, \sigma_2 = 0.2$, $T = 1$ (year), $\theta = 1.0$ and $\rho = 0.4$. We assume that the functions h_i $(i = 1, 2)$ are specified by $\tau_i = h_i^{-1}(Z_i) := -\log\left(\Phi(-Z_i)\right)/\bar{\lambda}_i$ with parameters $\bar{\lambda}_1 = 0.01$ and $\bar{\lambda}_2 = 0.02$, respectively, where Φ denotes the standard normal distribution function. Such a specification of h_i follows from the naive assumption that the default time τ_i follows the exponential distribution with constant hazard rate $\bar{\lambda}_i$, namely, $\mathbb{P}(\tau_i > t) = \exp(-\bar{\lambda}_i t)$.

The two trajectories in Figure 1 demonstrate that $\kappa_t^{(1)}$ moves simultaneously in response to the increase or decrease of $\lambda_t^{(1)}$. This property arises because both $\kappa_t^{(1)}$ and $\lambda_t^{(1)}$ are functionals of the same form of information flows ξ_t^1 and ξ_t^2, except for the difference in the weighted average

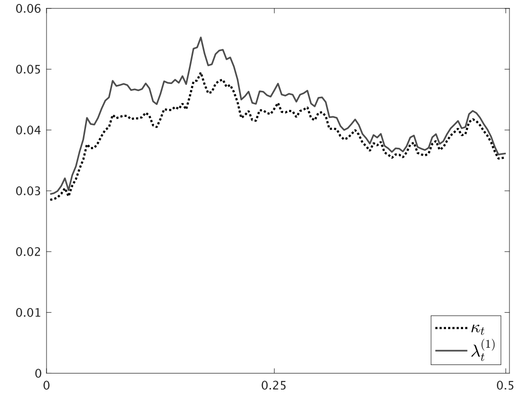

Figure 1: Simulated sample trajectories of κ_t and $\lambda_t^{(1)}$ for reference portfolio $\{1, 2\}$.

by B_u^{-1} across future time points $u \geq t$. And the difference between $\lambda_t^{(1)}$ and $\kappa_t^{(1)}$ decreases as time t approaches the maturity T, attributed to the gradual fading of the influence of the weighted average by B_u^{-1}.

4 Main Results

This section aims to investigate in more detail the semimaritingale decomposition representation (7) satisfied by the T-maturity market FtDS spread $\kappa_t^{(1)}$ derived in Theorem 2.1. Specifically, under the information-based model, we explicitly derive the stochastic differential equation satisfied by the T-maturity market FtDS spread $\kappa_t^{(1)}$.

First, similar to the discussions in Nakagawa and Takada [15] before stating their main results, we will focus on the conditional joint survival probabilities and related quantities. Thus, we define some $\{\mathcal{F}_t\}$-adapted processes needed to present the main results as follows.

For $x_1, \ldots, x_n \in \mathbb{R}$ and $i \in [n]$, we set

$$\begin{aligned} \zeta_t &:= \int_{\mathbb{R}^n} p_0\left((z_j)_{j\in[n]}\right) \prod_{j\in[n]} \mathcal{E}_t(\xi^j; z_j)(dz_j)_{j\in[n]}, \\ \zeta_{t,i} &:= \int_{\mathbb{R}^n} z_i p_0\left((z_j)_{j\in[n]}\right) \prod_{j\in[n]} \mathcal{E}_t(\xi^j; z_j)(dz_j)_{j\in[n]}, \end{aligned} \tag{9}$$

$$\varphi_t(x_1, \ldots, x_n) := \int_{\mathbb{R}^n} \prod_{j\in[n]} \mathbf{1}_{\{z_j > x_j\}} p_0\left((z_j)_{j\in[n]}\right) \prod_{j\in[n]} \mathcal{E}_t(\xi^j; z_j)(dz_j)_{j\in[n]},$$

$$\varphi_{t,i}(x_1, \ldots, x_n) := \int_{\mathbb{R}^n} \prod_{j\in[n]} \mathbf{1}_{\{z_j > x_j\}} z_i p_0\left((z_j)_{j\in[n]}\right) \prod_{j\in[n]} \mathcal{E}_t(\xi^j; z_j)(dz_j)_{j\in[n]},$$

$$
\begin{aligned}
&\varphi(u \mid t) := \varphi_t(h_1(u), \dots, h_n(u)), \quad \varphi_i(u \mid t) := \varphi_{t,i}(h_1(u), \dots, h_n(u)), \\
&\psi(u \mid t) := -\sum_{j=1}^{n} \partial_j \varphi(u \mid t), \quad \psi_i(u \mid t) := -\sum_{j=1}^{n} \partial_j \varphi_i(u \mid t).
\end{aligned} \tag{10}
$$

Here $\partial_j \varphi(u \mid t)$ and $\partial_j \varphi_i(u \mid t)$ appeared in (10) are specified as below.

$$
\begin{aligned}
\partial_j \varphi(u \mid t) &= \frac{\partial}{\partial u_j} \varphi_t(h_1(u_1), \dots, h_n(u_n))|_{u_1 = \cdots = u_n = u} \\
&= h_j'(u) \frac{\partial}{\partial x_j} \varphi_t(x_1, \dots, x_n))\Big|_{x_\ell = h_\ell(u)\ (\ell \in [n])} \\
&= h_j'(u) \frac{\partial}{\partial x_j} \int_{\mathbb{R}^n} \prod_{\ell \in [n]} \mathbf{1}_{\{z_\ell > x_\ell\}} p_0\left((z_\ell)_{\ell \in [n]}\right) \\
&\qquad\qquad \times \prod_{\ell \in [n]} \mathcal{E}_t(\xi^\ell; z_\ell)(dz_\ell)_{\ell \in [n]}\Big|_{x_\ell = h_\ell(u)\ (\ell \in [n])} \\
&= -h_j'(u) \int_{\mathbb{R}^{n-1}} \prod_{\ell \in [n] \setminus \{j\}} \mathbf{1}_{\{z_\ell > h_\ell(u)\}} p_0\left(z_1, \dots, z_{j-1}, h_j(u), z_{j+1}, \dots, z_n\right) \\
&\qquad\qquad \times \mathcal{E}_t(\xi^j; h_j(u)) \prod_{\ell \in [n] \setminus \{j\}} \mathcal{E}_t(\xi^\ell; z_\ell)(dz_\ell)_{\ell \in [n] \setminus \{j\}}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
&\partial_j \varphi_i(u \mid t) \\
&= h_j'(u) \frac{\partial}{\partial x_j} \int_{\mathbb{R}^n} \prod_{\ell \in [n]} \mathbf{1}_{\{z_\ell > x_\ell\}} z_i p_0\left((z_\ell)_{\ell \in [n]}\right) \prod_{\ell \in [n]} \mathcal{E}_t(\xi^\ell; z_\ell)(dz_\ell)_{\ell \in [n]}\Big|_{x_\ell = h_\ell(u)\ (\ell \in [n])} \\
&= \begin{cases}
-h_j'(u) \displaystyle\int_{\mathbb{R}^{n-1}} \prod_{\ell \in [n] \setminus \{j\}} \mathbf{1}_{\{z_\ell > h_\ell(u)\}} z_i p_0\left(z_1, \dots, z_{j-1}, h_j(u), z_{j+1}, \dots, z_n\right) & \\
\qquad\qquad \times \mathcal{E}_t(\xi^j; h_j(u)) \prod_{\ell \in [n] \setminus \{j\}} \mathcal{E}_t(\xi^\ell; z_\ell)(dz_\ell)_{\ell \in [n] \setminus \{j\}} & \text{if } j \neq i \\
& \\
-h_i'(u) \displaystyle\int_{\mathbb{R}^{n-1}} \prod_{\ell \in [n] \setminus \{i\}} \mathbf{1}_{\{z_\ell > h_\ell(u)\}} h_i(u) p_0\left(z_1, \dots, z_{i-1}, h_i(u), z_{i+1}, \dots, z_n\right) & \\
\qquad\qquad \times \mathcal{E}_t(\xi^i; h_i(u)) \prod_{\ell \in [n] \setminus \{i\}} \mathcal{E}_t(\xi^\ell; z_\ell)(dz_\ell)_{\ell \in [n] \setminus \{i\}} & \text{if } j = i
\end{cases}
\end{aligned}
$$

The process ζ_t serves to normalize other quantities so that their ratios to ζ_t can be interpreted as conditional probabilities or conditional expectations. As direct consequences of Proposition 3.1, we have

$$
\mathbb{E}[Z_i \mid \mathcal{F}_t] = \frac{\zeta_{t,i}}{\zeta_t}, \quad \bar{F}_{(1)}(u; t) = \mathbb{P}(\tau_1 > u, \dots, \tau_n > u \mid \mathcal{F}_t) = \frac{\varphi(u \mid t)}{\zeta_t},
$$

$$
\begin{aligned}
&\mathbb{E}[\mathbf{1}_{\{\tau_{(1)}>u\}}Z_i \mid \mathcal{F}_t] = \mathbb{E}[\mathbf{1}_{\{\tau_1>u,\ldots,\tau_n>u\}}Z_i \mid \mathcal{F}_t] = \frac{\varphi_i(u \mid t)}{\zeta_t}, \\
&f_{(1)}(u;t) = -\frac{d}{du}\bar{F}_{(1)}(u;t) \\
&\qquad\qquad = -\sum_{j=1}^n \partial_j \mathbb{P}(\tau_1 > u, \ldots, \tau_n > u \mid \mathcal{F}_t)\Big|_{z_j=h_j(u)} = \frac{\psi(u \mid t)}{\zeta_t}, \\
&-\sum_{j=1}^n \partial_j \mathbb{E}[\mathbf{1}_{\{\tau_{(1)}>u\}}Z_i \mid \mathcal{F}_t]\Big|_{z_j=h_j(u)} \\
&\qquad\qquad = -\sum_{j=1}^n \partial_j \mathbb{E}[\mathbf{1}_{\{\tau_1>u,\ldots,\tau_n>u\}}Z_i \mid \mathcal{F}_t]\Big|_{z_j=h_j(u)} = \frac{\psi_i(u \mid t)}{\zeta_t}.
\end{aligned}
$$

In addition, we define the followings for calculating the market FtDS spread. For $i \in [n]$,

$$
\begin{aligned}
\Phi_t(T) &:= \int_t^T B_u^{-1}\varphi(u \mid t)du \\
&= \int_t^T B_u^{-1} \int_{\mathbb{R}^n} \prod_{j\in[n]} \mathbf{1}_{\{z_j>x_j\}} p_0\left((z_j)_{j\in[n]}\right) \prod_{j\in[n]} \mathcal{E}_t(\xi^j;z_j)(dz_j)_{j\in[n]}du,
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\Phi_{t,i}(T) &:= \int_t^T B_u^{-1}\varphi_i(u \mid t)du \\
&= \int_t^T B_u^{-1} \int_{\mathbb{R}^n} \prod_{j\in[n]} \mathbf{1}_{\{z_j>x_j\}} z_i p_0\left((z_j)_{j\in[n]}\right) \prod_{j\in[n]} \mathcal{E}_t(\xi^j;z_j)(dz_j)_{j\in[n]}du,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\Psi_t(T) &:= \int_t^T B_u^{-1}\psi(u \mid t)du \\
&= -\int_t^T B_u^{-1} \sum_{j=1}^n \partial_j \Big[\int_{\mathbb{R}^n} \prod_{j\in[n]} \mathbf{1}_{\{z_j>x_j\}} p_0\left((z_j)_{j\in[n]}\right) \\
&\qquad\qquad \times \prod_{j\in[n]} \mathcal{E}_t(\xi^j;z_j)(dz_j)_{j\in[n]}\Big]\Bigg|_{z_j=h_j(u)} du
\end{aligned}
\tag{13}
$$

$$
\begin{aligned}
\Psi_{t,i}(T) &:= \int_t^T B_u^{-1}\psi_i(u \mid t)du \\
&= -\int_t^T B_u^{-1} \sum_{j=1}^n \partial_j \Big[\int_{\mathbb{R}^n} \prod_{j\in[n]} \mathbf{1}_{\{z_j>x_j\}} z_i p_0\left((z_j)_{j\in[n]}\right) \\
&\qquad\qquad \times \prod_{j\in[n]} \mathcal{E}_t(\xi^j;z_j)(dz_j)_{j\in[n]}\Big]\Bigg|_{z_j=h_j(u)} du.
\end{aligned}
\tag{14}
$$

At last, we define another process $\{W_t^i\}_{i\in[n]}$ as

$$W_t^i \ := \ \xi_t^i - \sigma_i \int_0^t \frac{\zeta_{u,i}}{\zeta_u} du \ = \ \xi_t^i - \sigma_i \int_0^t \mathbb{E}[Z_i \mid \mathcal{F}_u] du. \tag{15}$$

It follows from Levy's characterization theorem that the process $\{W_t^i\}_{i\in[n]}$ are $\{\mathcal{F}_t\}$-standard Brownian motions. We aim to represent the stochastic differential equations with respect to the $\{\mathcal{F}_t\}$-standard Brownian motions $\{W_t^i\}_{i\in[n]}$.

We should remark that the ratios of the above variables to the normalizer ζ_t can be regarded such as

$$\frac{\Phi_t(T)}{\zeta_t} = \int_t^T B_u^{-1} \mathbb{P}(\tau_1 > u, \tau_2 > u \mid \mathcal{F}_t) du.$$

We also mention that when comparing the case of defaultable bonds discussed in [15], it differs due to the weighted integral with respect to B_u^{-1} because of the infinitesimal cash flow payment of FtDS spread.

Now we present one of the main results: the stochastic differential equation satisfied by the T-maturity market FtDS spread $\kappa_t^{(1)}$ under the information-based model.

Proposition 4.1.

$$\begin{aligned} d\kappa_t^{(1)} = \mathbf{1}_{\{t<\tau_{(1)}\}} \kappa_t^{(1)} \Bigg[& -\sum_{i=1}^n \sigma_i \left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\Psi_{t,i}(T)}{\Psi_t(T)} \right) dW_t^i \\ & + \sum_{i=1}^n \sigma_i^2 \left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\Psi_{t,i}(T)}{\Psi_t(T)} \right) \left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right) dt \\ & + \frac{B_t^{-1} \bar{F}_{(1)}(t;t)}{\theta \int_t^T B_u^{-1} \mathbb{E}\left[\lambda_u^{(1)} \bar{F}_{(1)}(u;u) \mid \mathcal{F}_t \right] du} \left(\kappa_t^{(1)} - \theta \lambda_t^{(1)} \right) dt \Bigg] \\ + \kappa_{t-}^{(1)} d\mathbf{1}_{\{t<\tau_{(1)}\}}. \end{aligned} \tag{16}$$

We give the proof of the proposition in the next section.

As for the expression $\kappa_t^{(1)} - \theta\lambda_t^{(1)}$ in the last drift term, we can see on the event $\{t < \tau_{(1)}\}$,

$$\begin{aligned} & \frac{\kappa_t^{(1)} B_t^{-1} \bar{F}_{(1)}(t;t)}{\theta \int_t^T B_u^{-1} \mathbf{E}\left[\lambda_u^{(1)} \bar{F}_{(1)}(u;u) \mid \mathcal{F}_t \right] du} \left(\kappa_t^{(1)} - \theta \lambda_t^{(1)} \right) \\ & = \frac{\kappa_t^{(1)} B_t^{-1} \bar{F}_{(1)}(t;t)}{\theta \int_t^T B_u^{-1} \mathbf{E}\left[\lambda_u^{(1)} \bar{F}_{(1)}(u;u) \mid \mathcal{F}_t \right] du} \left(\frac{\theta \int_t^T B_u^{-1} f_{(1)}(u;t) du}{\int_t^T B_u^{-1} \bar{F}_{(1)}(u;t) du} - \theta \lambda_t^{(1)} \right) \\ & = \frac{\kappa_t^{(1)} B_t^{-1} \bar{F}_{(1)}(t;t)}{\int_t^T B_u^{-1} \mathbb{E}\left[\bar{F}_{(1)}(u;u) \mid \mathcal{F}_t \right] du} \frac{\int_t^T B_u^{-1} \mathbb{E}\left[\left(\lambda_u^{(1)} - \lambda_t^{(1)} \right) \bar{F}_{(1)}(u;u) \mid \mathcal{F}_t \right] du}{\int_t^T B_u^{-1} \mathbf{E}\left[\lambda_u^{(1)} \bar{F}_{(1)}(u;u) \mid \mathcal{F}_t \right] du}. \end{aligned}$$

This implies that market FtDS spread can move upward if the first-to-default hazard rate is likely to increase ($\lambda_u^{(1)} - \lambda_t^{(1)} > 0$ for $u > t$), and vice versa. Indeed, this property does not depend on a specific mathematical model.

The last consideration seems to imply that in order to understand the dynamics of $\kappa_t^{(1)}$ more thoroughly, it is necessary to know the dynamics of $\lambda_t^{(1)}$. As such, we also seek the SDE satisfied by the first-to-default hazard rate process $\lambda_t^{(1)}$.

Proposition 4.2. *Let $\varphi(t) := \varphi(t \mid t), \varphi_i(t) := \varphi_i(t \mid t), \psi(t \mid t) := \psi(t)$ and $\psi_i(t) := \psi_i(t \mid t)$. Then we have*

$$\begin{aligned} d\lambda_t^{(1)} = \lambda_t^{(1)} \Bigg[& - \sum_{i=1}^n \sigma_i \left(\frac{\varphi_i(t)}{\varphi(t)} - \frac{\psi_i(t)}{\psi(t)} \right) dW_t^i \\ & + \sum_{i=1}^n \sigma_i^2 \left(\frac{\varphi_i(t)}{\varphi(t)} - \frac{\zeta_{t,i}}{\zeta_t} \right) \left(\frac{\varphi_i(t)}{\varphi(t)} - \frac{\psi_i(t)}{\psi(t)} \right) dt \\ & + \lambda_t^{(1)} dt + \sum_{i=1}^n \frac{\partial_i \psi(t)}{\psi(t)} dt \Bigg]. \end{aligned} \tag{17}$$

The proof is given in Appendix A.

Comparing (16) and (17), it is apparent that the volatility term and a part of the drift term of the market FtDS spread $\kappa_t^{(1)}$ and the first-to-default hazard rate $\lambda_t^{(1)}$ have a similar structure because $\Phi_t(T)$ and $\Psi_t(T)$ indeed corresponds to $\varphi(t)$ and $\psi(t)$ respectively.
As for the volatility terms of $\kappa_t^{(1)}$, we can see

$$\sigma_i \left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\Psi_{t,i}(T)}{\Psi_t(T)} \right) = \sigma_i \left(\frac{\int_t^T B_u^{-1} \varphi_i(u \mid t) du}{\int_t^T B_u^{-1} \varphi(u \mid t) du} - \frac{\int_t^T B_u^{-1} \psi_i(u \mid t) du}{\int_t^T B_u^{-1} \psi(u \mid t) du} \right).$$

Moreover, the volatility terms of $\lambda_t^{(1)}$ can be viewed as

$$\begin{aligned} \sigma_i \left(\frac{\varphi_i(t)}{\varphi(t)} - \frac{\psi_i(t)}{\psi(t)} \right) &= \sigma_i \left(\frac{\varphi_i(t \mid t)}{\varphi(t \mid t)} - \frac{\psi_i(t \mid t)}{\psi(t \mid t)} \right) \\ &= \sigma_i \left(\frac{\mathbb{E}[\mathbf{1}_{\{\tau_{(1)} > t\}} Z_i \mid \mathcal{F}_t]}{\mathbb{P}(\tau_{(1)} > t \mid \mathcal{F}_t)} - \frac{-\sum_{j=1}^n \partial_j \mathbb{E}[\mathbf{1}_{\{\tau_{(1)} > t\}} Z_i \mid \mathcal{F}_t] \Big|_{z_j = h_j(t)}}{-\sum_{j=1}^n \partial_j \mathbb{P}(\tau_{(1)} > t \mid \mathcal{F}_t) \Big|_{z_j = h_j(t)}} \right) \\ &= \sigma_i \left(\frac{\mathbb{E}[\mathbf{1}_{\{\tau_{(1)} > t\}} Z_i \mid \mathcal{F}_t]}{\bar{F}_{(1)}(t;t)} - \frac{-\sum_{j=1}^n \partial_j \mathbb{E}[\mathbf{1}_{\{\tau_{(1)} > t\}} Z_i \mid \mathcal{F}_t] \Big|_{z_j = h_j(t)}}{f_{(1)}(t;t)} \right) \\ &= \sigma_i \left(\mathbb{E}[Z_i \mid \mathcal{F}_t, \tau_{(1)} > t] - \sum_{j=1}^n \mathbb{E}[Z_i \mid \mathcal{F}_t, \tau_{(1)} = \tau_j = t] \right). \end{aligned}$$

These findings imply that the noise impact of the i-th information process on the dynamics of $\kappa_t^{(1)}$ as well as $\lambda_t^{(1)}$ can be explained in terms of the information quality of the survival condition in the estimation of the i-th credit factor Z_i, which is measured via the discrepancy between the $\mathcal{F}_t$-conditional expectation of the i-th credit factor conditioned on $\tau_{(1)} > t$ and $\tau_{(1)} = t$.

Similarly, as for the following part of the second term in (17), we have

$$\frac{\varphi_i(t)}{\varphi(t)} - \frac{\zeta_{t,i}}{\zeta_t} = \mathbb{E}[Z_i \mid \mathcal{F}_t, \tau_{(1)} > t] - \mathbb{E}[Z_i \mid \mathcal{F}_t].$$

Hence, we find that the drift term of the dynamics of $\kappa_t^{(1)}$ as well as $\lambda_t^{(1)}$ is also influenced by the information quality given by the difference between the $\mathcal{F}_t$-conditional expectation of the i-th credit factor conditioned on $\tau_{(1)} > t$ and unconditioned on $\tau_{(1)}$ in the estimation of the i-th credit factor.

5 Proof of Proposition 4.1

Remember the stochastic differential equation (7) in Theorem 2.1 which is in general satisfied by the market FtDS spread process $\{\kappa_t^{(1)}\}$ with maturity T. From Proposition 3.1, it follows that the $\{\mathcal{F}_t\}$-martingales m_t^A (defined in (3)) and m_t^θ (defined in (4)) appearing in (7) can be rewritten as

$$\begin{aligned} m_t^A &= \int_t^T B_u^{-1} \bar{F}_{(1)}(u;t) du + \int_0^t B_u^{-1} \bar{F}_{(1)}(u;u) du \\ &= \frac{\Phi_t(T)}{\zeta_t} + \int_0^t B_u^{-1} \bar{F}_{(1)}(u;u) du, \end{aligned} \tag{18}$$

and

$$\begin{aligned} m_t^\theta &= \int_t^T B_u^{-1} f_{(1)}(u;t) du + \int_0^t B_u^{-1} f_{(1)}(u;u) du \\ &= \frac{\Psi_t(T)}{\zeta_t} + \int_0^t B_u^{-1} f_{(1)}(u;u) du. \end{aligned} \tag{19}$$

5.1 Stochastic differentiation of the martingale m_t^A

We first examine the term $\dfrac{\Phi_t(T)}{\zeta_t}$ that appears in (18). Since we see for each i,

$$d\mathcal{E}_t(\xi^i; z_i) = \sigma_i z_i \exp\left(\sigma_i z_i \xi_t^i - \frac{t}{2}\sigma_i^2 z_i^2\right) d\xi_t^i,$$

it follows from Ito's formula on $\Phi_t(T)$ in (11) that

$$d\Phi_t(T) = -B_t^{-1}\zeta_t \bar{F}_{(1)}(t;t)dt + \sum_{i=1}^n \sigma_i \Phi_{t,i}(T) d\xi_t^i,$$

where $\Phi_{t,i}(T)$ is defined in (12). And also applying Ito's formula to ζ_t, we have

$$d\zeta_t = \sum_{i=1}^n \sigma_i \zeta_{t,i} d\xi_t^i, \tag{20}$$

where $\zeta_{t,i}$ is defined in (9). Thus by applying Ito's formula again, we have

$$dm_t^A = \frac{\Phi_t(T)}{\zeta_t}\left\{\frac{-B_t^{-1}\zeta_t\bar{F}_{(1)}(t;t)dt + \sum_{i=1}^n \sigma_i\Phi_{t,i}(T)d\xi_t^i}{\Phi_t(T)} - \frac{\sum_{i=1}^n \sigma_i\zeta_{t,i}d\xi_t^i}{\zeta_t}\right.$$
$$\left. + \frac{\sum_{i=1}^n \sigma_i^2\zeta_{t,i}^2}{\zeta_t^2}dt - \frac{\sum_{i=1}^n \sigma_i^2\Phi_{t,i}(T)\zeta_{t,i}}{\Phi_t(T)\zeta_t}dt\right\} + B_t^{-1}\bar{F}_{(1)}(t;t)dt$$
$$= \frac{\Phi_t(T)}{\zeta_t}\sum_{i=1}^n \sigma_i\left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t}\right)\left(d\xi_t^i - \sigma_1\frac{\zeta_{t,i}}{\zeta_t}dt\right).$$

Therefore, if we use the $\{\mathcal{F}_t\}$-standard Brownian motion W_t^i given in (15) instead of ξ_t^i, we can conclude that

$$dm_t^A = \frac{\Phi_t(T)}{\zeta_t}\sum_{i=1}^n \sigma_i\left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t}\right)dW_t^i.$$

5.2 Stochastic differentiation of the martingale m_t^θ

Next, as for dm_t^θ, we compute the term $\frac{\Psi_t(T)}{\zeta_t}$ that appears in (19). Applying Ito's formula to $\Psi_t(T)$ in (13),

$$d\Psi_t(T) = -B_t^{-1}\zeta_t f_{(1)}(t;t)dt + \sum_{i=1}^n \sigma_i\Psi_{t,i}(T)d\xi_t^i,$$

where $\Psi_{t,i}(T)$ is defined in (14).

Hence applying Ito's formula to the quotient $\frac{\Psi_t(T)}{\zeta_t}$, we obtain

$$dm_t^\theta = \frac{\Psi_t(T)}{\zeta_t}\left\{-\frac{B_t^{-1}\zeta_t f_{(1)}(t;t)dt + \sum_{i=1}^n \sigma_i\Psi_{t,i}(T)d\xi_t^i}{\Psi_t(T)} - \frac{\sum_{i=1}^n \sigma_i\zeta_{t,i}d\xi_t^i}{\zeta_t}\right.$$
$$\left. + \frac{\sum_{i=1}^n \sigma_i^2\zeta_{t,i}^2}{\zeta_t^2}dt - \frac{\sum_{i=1}^n \sigma_i\Psi_{t,i}(T)\zeta_{t,i}}{\Psi_t(T)\zeta_t}dt\right\} + B_t^{-1}f_{(1)}(t;t)dt$$
$$= \frac{\Psi_t(T)}{\zeta_t}\sum_{i=1}^n \sigma_i\left(\frac{\Psi_{t,i}(T)}{\Psi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t}\right)dW_t^i.$$

5.3 Stochastic dynamics of $\kappa_t^{(1)}$

From the above results, we can compute $d\langle m^A, m^A\rangle_t$ and $d\langle m^A, m^\theta\rangle_t$ in (7) as follows.

$$d\langle m^A, m^A\rangle_t = \left(\frac{\Phi_t(T)}{\zeta_t}\right)^2\sum_{i=1}^n \sigma_i^2\left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t}\right)^2 dt,$$
$$d\langle m^A, m^\theta\rangle_t = \frac{\Phi_t(T)\Psi_t(T)}{\zeta_t^2}\sum_{i=1}^n \sigma_i^2\left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t}\right)\left(\frac{\Psi_{t,i}(T)}{\Psi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t}\right)dt.$$

Therefore we can conclude that the SDE (7) for the market FtDS spread process $\{\kappa_t^{(1)}\}$ can be represented by

$$
\begin{aligned}
d\kappa_t^{(1)} = \mathbf{1}_{\{t<\tau_{(1)}\}} \frac{1}{A(t,T)} \Bigg[& B_t^{-1} \bar{F}_{(1)}(t;t) \left(\kappa_t^{(1)} - \theta \lambda_t^{(1)} \right) dt \\
& + \frac{1}{A(t,T)} \Bigg\{ \kappa_t^{(1)} \left(\frac{\Phi_t(T)}{\zeta_t} \right)^2 \sum_{i=1}^n \sigma_i^2 \left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right)^2 dt \\
& - \theta \frac{\Phi_t(T)\Psi_t(T)}{\zeta_t^2} \sum_{i=1}^n \sigma_i^2 \left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right) \left(\frac{\Psi_{t,i}(T)}{\Psi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right) dt \Bigg\} \\
& + \sum_{i=1}^n \sigma_i \Big\{ \theta \frac{\Psi_t(T)}{\zeta_t} \left(\frac{\Psi_{t,i}(T)}{\Psi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right) - \kappa_t^{(1)} \frac{\Phi_t(T)}{\zeta_t} \left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right) \Big\} dW_t^i \Bigg] \\
& + \kappa_{t-}^{(1)} d\mathbf{1}_{\{t<\tau_{(1)}\}}.
\end{aligned}
$$

We have thus derived the explicit form of the stochastic differential equation for $\{\kappa_t^{(1)}\}$. However, the coefficients seem still too complicated. From the relations

$$
A(t,T) = \frac{\Phi_t(T)}{\zeta_t}, \quad \theta(t,T) = \theta \frac{\Psi_t(T)}{\zeta_t}, \quad \kappa_t^{(1)} = \mathbf{1}_{\{t<\tau_{(1)}\}} \frac{\theta(t,T)}{A(t,T)},
$$

it follows that the diffusion term is further calculated as follows.

$$
\begin{aligned}
& \frac{1}{A(t,T)} \sum_{i=1}^n \sigma_i \Big\{ \theta \frac{\Psi_t(T)}{\zeta_t} \left(\frac{\Psi_{t,i}(T)}{\Psi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right) - \kappa_t^{(1)} \frac{\Phi_t(T)}{\zeta_t} \left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right) \Big\} dW_t^i \\
& = \kappa_t^{(1)} \sum_{i=1}^n \sigma_i \Big\{ \left(\frac{\Psi_{t,i}(T)}{\Psi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right) - \left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right) \Big\} dW_t^i.
\end{aligned}
$$

Via a similar argument, we rewrite the following part of the drift term as follows.

$$
\begin{aligned}
\frac{1}{A(t,T)} \Bigg\{ & \kappa_t^{(1)} \left(\frac{\Phi_t(T)}{\zeta_t} \right)^2 \sum_{i=1}^n \sigma_i^2 \left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right)^2 dt \\
& - \theta \frac{\Phi_t(T)\Psi_t(T)}{\zeta_t^2} \sum_{i=1}^n \sigma_i^2 \left(\frac{\Phi_{t,i}(T)}{\Phi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right) \left(\frac{\Psi_{t,i}(T)}{\Psi_t(T)} - \frac{\zeta_{t,i}}{\zeta_t} \right) dt \Bigg\}.
\end{aligned}
$$

Since $f_{(1)}(u;u) = \lambda_u^{(1)} \bar{F}_{(1)}(u;u)$, we can see on the set $\{t < \tau_{(1)}\}$,

$$
\begin{aligned}
\frac{1}{A(t,T)} B_t^{-1} \bar{F}_{(1)}(t;t) &= \frac{\theta(t,T)}{A(t,T)} \frac{B_t^{-1} \bar{F}_{(1)}(t;t)}{\theta(t,T)} \\
&= \kappa_t^{(1)} \frac{B_t^{-1} \bar{F}_{(1)}(t;t)}{\theta \int_t^T B_u^{-1} \mathbb{E} \left[\lambda_u^{(1)} \bar{F}_{(1)}(u;u) \mid \mathcal{F}_t \right] du}.
\end{aligned}
$$

These calculations finally imply the representation given in (16).

6 Conclusion

By applying the multi-asset credit risk model based on the information-based approach to analysis of the market FtDS spread process, we derived the SDE followed by the market FtDS spread process more specifically. In addition, we also derived the SDE for the first-to-default hazard rate process that appears within the SDE of the market FtDS spread and were able to observe the structural similarities between them via the volatility term and a part of drift term of the SDEs.

In particular, the market FtDS spread and the first-to-default hazard rate are likely to move in the same direction under the influence of the information flow, but their difference converges over time.

Furthermore, we found that the dynamics of the market FtDS spread as well as the first-to-default hazard rate can be explained in terms of the information quality of the survival condition in the estimation of each credit-related individual factor, which is measured via the discrepancy between a couple of conditional expectations of each credit factor with respect to different conditions related to the first default time.

We believe that the multi-name information-based credit model has various other potential applications, which are promising areas for future research. For example, it seems useful for theoretically analyzing the dynamics of the market spread of Credit Default Index Swaps, such as iTraxx and CDX. As a practical application, the multi-name information-based credit model enables us to discuss the effectiveness of delta hedging strategies for multi-name credit derivatives like FtDS using single-name CDSs. Additionally, it allows for stress scenario analysis of the present value of some CDS portfolios, paying attention to wrong-way risk.

A Proof of Proposition 4.2

To begin with, it follows from the first-to-default hazard rate process $\{\lambda_t^{(1)}\}$ defined in (1)

$$
\begin{aligned}
d\lambda_t^{(1)} &= d\left(\frac{f_{(1)}(t;t)}{\bar{F}_{(1)}(t;t)}\right) \\
&= \frac{f_{(1)}(t;t)}{\bar{F}_{(1)}(t;t)}\left(\frac{df_{(1)}(t;t)}{f_{(1)}(t;t)} - \frac{d\bar{F}_{(1)}(t;t)}{\bar{F}_{(1)}(t;t)} + \frac{d\langle \bar{F}_{(1)}(\cdot;\cdot)\rangle_t}{\bar{F}_{(1)}(t;t)^2} - \frac{d\langle f_{(1)}(\cdot;\cdot), \bar{F}_{(1)}(\cdot;\cdot)\rangle_t}{f_{(1)}(t;t)\bar{F}_{(1)}(t;t)}\right).
\end{aligned} \tag{21}
$$

In order to derive the SDE followed by $\lambda_t^{(1)}$, we need the SDEs followed by $f_{(1)}(t;t)$ and $\bar{F}_{(1)}(t;t)$ respectively.

We note that $\bar{F}_{(1)}(t;t)$ can be represented by

$$
\bar{F}_{(1)}(t;t) = \frac{\varphi(t)}{\zeta_t} \quad \left(= \frac{\displaystyle\int_{\mathbb{R}^n} \prod_{j\in[n]} \mathbf{1}_{\{z_j > h_j(t)\}} p_0\left((z_j)_{j\in[n]}\right) \prod_{j\in[n]} \mathcal{E}_t(\xi^j; z_j)(dz_j)_{j\in[n]}}{\displaystyle\int_{\mathbb{R}^n} p_0\left((z_j)_{j\in[n]}\right) \prod_{j\in[n]} \mathcal{E}_t(\xi^j; z_j)(dz_j)_{j\in[n]}} \right).
$$

From (20) we remember $d\zeta_t = \sum_{i=1}^{n} \sigma_i \zeta_{t,i} d\xi_t^i$. In addition, we have

$$d\varphi(t) = \sum_{i=1}^{n} \left\{ \partial_i \varphi(t) + \sigma_i \varphi(t) d\xi_t^i \right\} = -\psi(t)dt + \sum_{i=1}^{n} \sigma_i \varphi(t) d\xi_t^i.$$

Thus we have

$$\begin{aligned}
d\bar{F}_{(1)}(t;t) = d\left(\frac{\varphi(t)}{\zeta_t}\right) &= \bar{F}_{(1)}(t;t)\left(\frac{d\varphi(t)}{\varphi(t)} - \frac{d\zeta_t}{\zeta_t} + \frac{d\langle\zeta\rangle_t}{(\zeta_t)^2} - \frac{d\langle\varphi,\zeta\rangle_t}{\varphi(t)\zeta_t}\right) \\
&= \bar{F}_{(1)}(t;t)\left\{\sum_{i=1}^{n} \sigma_i \left(\frac{\varphi_i(t)}{\varphi(t)} - \frac{\zeta_{t,i}}{\zeta_t}\right) d\xi_t^i - \sum_{i=1}^{n} \sigma_i^2 \frac{\zeta_{t,i}}{\zeta_t}\left(\frac{\varphi_i(t)}{\varphi(t)} - \frac{\zeta_{t,i}}{\zeta_t}\right) dt - \frac{\psi(t)}{\varphi(t)} dt\right\} \\
&= \bar{F}_{(1)}(t;t) \sum_{i=1}^{n} \sigma_i \left(\frac{\varphi_i(t)}{\varphi(t)} - \frac{\zeta_{t,i}}{\zeta_t}\right)\left(d\xi_t^i - \sigma_i \frac{\zeta_{t,i}}{\zeta_t} dt\right) - \bar{F}_{(1)}(t;t)\frac{\psi(t)}{\varphi(t)} dt \\
&= \bar{F}_{(1)}(t;t) \sum_{i=1}^{n} \sigma_i \left(\frac{\varphi_i(t)}{\varphi(t)} - \frac{\zeta_{t,i}}{\zeta_t}\right) dW_t^i - f_{(1)}(t;t)dt
\end{aligned}$$

The last equality holds because we have

$$f_{(1)}(t;t) = \frac{\psi(t)}{\zeta_t} = \frac{\varphi(t)}{\zeta_t}\frac{\psi(t)}{\varphi(t)} = \bar{F}_{(1)}(t;t)\frac{\psi(t)}{\varphi(t)}.$$

Next we move to the dynamics of $f_{(1)}(t;t)$. Note that

$$d\psi(t) = \sum_{i=1}^{n} \left\{ \partial_i \psi(t) + \sigma_i \psi_i(t) d\xi_t^i \right\} = \sum_{i=1}^{n} \partial_i \psi(t) dt + \sum_{i=1}^{n} \sigma_i \psi_i(t) d\xi_t^i.$$

Therefore we have

$$\begin{aligned}
df_{(1)}(t;t) = d\left(\frac{\psi(t)}{\zeta_t}\right) &= f_{(1)}(t;t)\left(\frac{d\psi(t)}{\psi(t)} - \frac{d\zeta_t}{\zeta_t} + \frac{d\langle\zeta\rangle_t}{(\zeta_t)^2} - \frac{d\langle\psi,\zeta\rangle_t}{\psi(t)\zeta_t}\right) \\
&= f_{(1)}(t;t)\left\{\sum_{i=1}^{n} \sigma_i \left(\frac{\psi_i(t)}{\psi(t)} - \frac{\zeta_{t,i}}{\zeta_t}\right) d\xi_t^i - \sum_{i=1}^{n} \sigma_i \frac{\zeta_{t,i}}{\zeta_t}\left(\frac{\psi_i(t)}{\psi(t)} - \frac{\zeta_{t,i}}{\zeta_t}\right) dt + \sum_{i=1}^{n} \frac{\partial_i \psi(t)}{\psi(t)} dt\right\} \\
&= f_{(1)}(t;t)\left\{\sum_{i=1}^{n} \sigma_i \left(\frac{\psi_i(t)}{\psi(t)} - \frac{\zeta_{t,i}}{\zeta_t}\right) dW_t^i + \sum_{i=1}^{n} \frac{\partial_i \psi(t)}{\psi(t)} dt\right\}
\end{aligned}$$

Finally, substituting these into (21) leads to the SDE (17).

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