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A Novel Extension of the Merton Model for Default Risk Assessment in Firms with Non-Market-Traded Operational Assets

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Abstract

In this paper, we present a novel structural credit risk model that is based on the Merton model to address the challenge of assessing default risk for firms with non-market-traded operational assets. Specifically, we introduce a model where the firm's average temporal revenue and operating expenses are represented as time integrals of an increasing function of depreciating operational assets, with the actual rates of change fluctuating stochastically. Using this setup, we discuss how to determine the debt face value and corresponding credit spread within this framework, and we outline a numerical method for these calculations based on the Merton model. Additionally, we introduce another model in which the uncertainty of the firm's total profit is assumed to follow Johnson's SU distribution, and we suggest the firm's debt valuation procedure for this alternative approach. We demonstrate some numerical experimental results for the two models and discuss the properties of these models in terms of assessing default risk.

Keywords: structural approach, the Merton model, debt valuation, non-market-traded operational assets, Johnson's SU distribution.

JEL Classification: C63, G32, G33

1 Introduction

The purpose of this study is to introduce a new structural-approach credit risk model that makes some improvements to the Merton [6] model, and to examine the issue of assessing the default risk of liabilities for firms with non-market-traded operational assets using the new model. Our model's novelty lies in the consideration of a situation where revenue and operating expenses (and consequently the profit, which is the difference between the two) depend on the value of operational assets while also being exposed to uncertainty.

Credit risk refers to the possibility that the borrower (debtor) may fall into default, causing the lender (creditor) to suffer financial losses due to the inability to receive repayment as agreed beforehand. For banking business of corporate lending, the assessment of the credit risk of borrowing firms has long been a crucial issue.

Among various approaches to modeling credit risk, we focus on the structural approach. It can be said that the structural approach was pioneered by Merton [6]. In Merton's model, a firm is assumed to finance with a discounted bond (debt) and equity to purchase the assets, and a default is defined as the state in which the company becomes insolvent at the time of debt repayment (in other words, the value of the firm's assets falls below the face value of its debt at that time). In

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addition, the stochastic dynamics of the firm's asset value is modeled using a geometric Brownian motion. Since the payoff of the discounted bond can be viewed as a short position in a European put option, where the firm's assets are the underlying asset and the face value of the debt is the strike price, the value of the discounted bond is calculated using the Black-Scholes-Merton framework, which had just been established at that time as the risk-neutral option pricing theory. As another representative example of the structural approach, well-known is the first passage time model, which defines the default time as the first time the stochastic process representing the firm value (or its proxy) reaches a threshold level triggering the default. (An early famous study on the first passage time model is Black and Cox [1].)

Merton model, though simple, can be considered an excellent model that offers various implications. Moreover, there are quite a few cases where the Merton model is applied to actual credit risk assessments in practice. However, its strong assumptions are quite far from the realities of most firms. One such assumption is that all assets are tradable in the market. This assumption enables the risk-neutral valuation of the firm's liabilities; however, the only type of firm where all assets are likely tradable would be an investment fund. Therefore, when the Merton model is applied to typical operating companies, the assumption that all assets are market-tradable is excessively strong.

Additionally, in typical businesses, the revenue is generated through products and services created using operational assets, and the profit is earned after deducting the operating expenses and so on. However, the Merton model does not explicitly study the cash flows of business activities, such as revenue or operating expenses. Some models that apply the structural approach using the earnings as a state variable are studied by Goldstein et al. [5] and Genser [4], but they do not delve into the perspectives of revenue and operating expenses.

We believe that considering revenue and operating expenses makes the model more complex. However, recent studies have emerged in which customers' deposit and withdrawal data of some banks are analyzed using statistical methods and/or machine learning techniques (e.g. Yamanaka and Yamamoto [8].) This suggests the possibility that analyzing deposit and withdrawal data could make it easier to model the stochastic dynamics of revenue and operating expenses.

Thus, while still somewhat simple at this stage, we propose a model in which the revenue and the operating expenses are dependent on the operational assets, and by considering some simple uncertainty, we extend the Merton model to clearly show that the change in the firm's balance sheet is driven by period profits.

The basic idea behind our modeling is largely inspired by the discussion in Section 1.3 "Non-tradable assets" of Capiński and Zastawniak [3], so our model has some similarities with theirs, although our focus and perspective are different form theirs.

2 First model with exponential martingales driven by two correlated Brownian motions

In this section, we introduce a model that describes a firm's balance sheet structure, the dynamics of its revenue and operating expenses, and debt valuation as an extension of the Merton model.

Note that some of the notation in this model references that used in the credit risk textbook by Capiński and Zastawniak [3].

To establish the foundation, let $(\Omega, \mathcal{F}, (\mathcal{F}_t), \mathbf{P})$ denote a filtered probability space, where $\mathbf{P}$ is the real probability measure.

2.1 Balance sheet

We consider a firm that begins some business at time 0 and liquidate it at a fixed time horizon $T \in (0, \infty)$. In addition, we suppose the followings on the firm.

- The firm requires operating assets valued at V_0^O and non-operating assets (viewed as cash) valued at V_0^{NO} at time 0, which are financed through debt and equity. Thus, the total value of the initial assets is given by $V_0 := V_0^O + V_0^{NO}$.
- The debt is raised by issuing a discounted bond with face value of F and maturity T, and the firm can actually borrow an amount of D_0 at time 0. The remaining $E_0 := V_0 - D_0$ is financed through the equity.
- At time T, the firm liquidates all of its assets, repays its debt, and distributes the remaining assets to the shareholders. However, there is a possibility that the firm may fall into "default," meaning that the liquidated assets may be insufficient to cover the face value of its debt.

We should remark that while the initial debt amount D_0 is initially given, the debt face value F is still unknown at this point. In other words, F is supposed to be a deterministic constant that will be determined later. Also, unknown is the credit yield of the discounted bond, denoted as k_D, introduced via the relation $D_0 = Fe^{-k_D T}$. Hence, our main problem of this paper is to derive an equation for obtaining the debt face value F through a discussion of debt valuation under a suitable model, and to numerically achieving F (or the credit yield k_D).

Then, we give the following assumptions on the dynamics of the assets over the period $[0, T)$.

- The operating assets depreciate over time at a constant depreciation rate $\eta > 0$ and ultimately lead to a liquidation value $V_0^O e^{-\eta T}$.
- The non-operating assets increase at the continuous compounded risk-free rate r.

For time $t \leq T$, the operating assets V_t^O and the non-operating assets V_t^{NO} are supposed to change deterministically in time and are respectively given by

$$V_t^O = V_0^O e^{-\eta t}, \ V_t^{NO} = V_0^{NO} e^{rt}.$$

2.2 Revenue, operation expenses, and profit

Next, we consider the dynamics of the revenue and the operating expenses to calculate the profit which is added to the assets at the time horizon. We denote by S_t and C_t stochastic processes that stand for the cumulative revenue and the cumulative operating expenses during the period $[0, t]$, respectively. So, the firm's profit during the period $[0, T]$ is given by $S_T - C_T$, the difference between the revenue and operating expenses during $[0, T]$. If the profit $S_T - C_T$ is positive, it is added to the assets at time T. On the other hand, if the profit is negative, the assets are reduced to cover the deficit amount.

We assume that both the expected cumulative revenue (resp. the expected cumulative operating expenses) are specified by the time-integral of a deterministic function of the instantaneous revenue (resp. the instantaneous operating expenses) at each point in time. We suppose that the functions of the instantaneous revenue and the instantaneous operating expenses depend only on the operating assets. Thus we denote by $s(v)$ and $c(v)$ some positive, increasing functions that represent the instantaneous revenue and instantaneous operating expenses, respectively[1].

It may seem like a strong assumption that $s(v)$ and $c(v)$ are determined solely by the operating assets at each point in time. However, rigorously estimating the temporal evolution of operating assets and their relationship with revenue and operating expenses is essential for formulating a long-term financial plan—an indispensable aspect of business management. Therefore, while other factors beyond operating assets may indeed exert some influence in practice, the assumption that operating assets are the primary determinants of revenue and operating expenses remains reasonably justified.

At the same time, uncertainties may lead to deviations from the long-term financial plan. Thus, it is natural to assume that actual revenue and operating expenses fluctuate stochastically over

[1] In the following discussion, it is essential to note that the revenue and the operating expenses expressed as time integrals are not random.

time around their respective expected values, which serve as long-term benchmarks for revenue and operating expenses.

Following the above discussion, we model the cumulative revenue S_t and cumulative operating expenses C_t as follows, incorporating uncertainty while ensuring that both are increasing processes.

$$S_t = \left(1+\kappa_S e^{\sigma_S W_t^S - \frac{(\sigma_S)^2}{2}t}\right)\int_0^t s(V_u^O)du = \left(1+\kappa_S e^{\sigma_S W_t^S - \frac{(\sigma_S)^2}{2}t}\right)\int_0^t s(V_0^O e^{-\eta u})du,$$
$$C_t = \left(1+\kappa_C e^{\sigma_C W_t^C - \frac{(\sigma_C)^2}{2}t}\right)\int_0^t c(V_u^O)du = \left(1+\kappa_C e^{\sigma_C W_t^C - \frac{(\sigma_C)^2}{2}t}\right)\int_0^t c(V_0^O e^{-\eta u})du,$$

where $\kappa_S, \kappa_C, \sigma_S$ and σ_C are positive parameters, and W_t^S and W_t^C are a couple of $(\mathcal{F}_t)$-standard Brownian motions with correlation parameter $\rho \in (-1, 1)$.

Since the stochastic process $e^{\sigma_\bullet W_t^\bullet - \frac{(\sigma_\bullet)^2}{2}t}$ is known as an exponential martingale, whose expectation is always one, so we have

$$\mathbf{E}^{\mathbf{P}}[S_t] = (1+\kappa_S)\int_0^t s(V_0^O e^{-\eta u})du, \quad \mathbf{E}^{\mathbf{P}}[C_t] = (1+\kappa_C)\int_0^t c(V_0^O e^{-\eta u})du.$$

Therefore, the terminal value of firm's assets V_T is given by

$$\begin{aligned} V_T &= V_T^O + V_T^{NO} + (S_T - C_T) \\ &= V_0^O e^{-\eta T} + V_0^{NO} e^{rT} + \left(1+\kappa_S e^{\sigma_S W_T^S - \frac{(\sigma_S)^2}{2}T}\right) I_S - \left(1+\kappa_C e^{\sigma_C W_T^C - \frac{(\sigma_C)^2}{2}T}\right) I_C, \end{aligned} \tag{1}$$

where we set

$$I_S := \int_0^T s(V_0^O e^{-\eta u})du, \quad I_C := \int_0^T c(V_0^O e^{-\eta u})du.$$

In particular, we note that the expectation of V_T is given by

$$\mathbf{E}^{\mathbf{P}}[V_T] = V_0^O e^{-\eta T} + V_0^{NO} e^{rT} + (1+\kappa_S)I_S - (1+\kappa_C)I_C. \tag{2}$$

The integral terms I_S and I_C included in firm's assets V_T can be calculated analytically depending on the integrands $s(v)$ and $c(v)$, and even if they cannot, they can be accurately evaluated using numerical integration.

2.3 Debt valuation

Whether the firm is into default or not at time T depends on the firm's whole asset V_T. If $V_T \geq F$, the debt holder can receive the face value F at time T. On the other hand, if $0 \leq V_T < F$, or the asset is not enough to cover the face value of the debt, the firm will default on its debt and the debt holder can only receive the asset value V_T. If $V_T < 0$, the firm cannot pay anything, so the payoff for the debt holder is zero.

At last, the debt payoff D_T at time T is specified by

$$D_T = F \cdot \mathbf{1}_{\{F \leq V_T\}} + V_T \cdot \mathbf{1}_{\{0 \leq V_T < F\}}.$$

Since we assume that the assets are not traded in the market, the so-called risk-neutral valuation cannot be applied for the debt valuation. Therefore, according to the idea of section 1.3 in Capiński and Zastawniak [3], we introduce the expected return on the debt μ_D over the period $[0, T]$ under $\mathbf{P}$ and assume the following relationship holds:

$$D_0 = \frac{\mathbf{E}^{\mathbf{P}}[D_T]}{1+\mu_D}.$$

In addition, if the market price of risk p on debt is introduced by

$$p = \frac{\mu_D - (e^{rT} - 1)}{\frac{1}{D_0}\sqrt{\mathrm{Var}^{\mathbf{P}}(D_T)}}.$$

Although we do not discuss the detailed estimation method for the market price of risk p on debt here, we believe that the most practical approach is to utilize historical data on bond prices issued by firms with a similar level of credit risk to the underlying firm. Specifically, this involves computing the mean and variance of such bond returns and estimating p accordingly based on the above formula.

Then we can achieve the following equation:

$$D_0 = e^{-rT}\left(\mathbf{E}^{\mathbf{P}}[D_T] - p\sqrt{\mathrm{Var}^{\mathbf{P}}(D_T)}\right) = e^{-rT}\left(\mathbf{E}^{\mathbf{P}}[D_T] - p\sqrt{\mathbf{E}^{\mathbf{P}}[(D_T)^2] - (\mathbf{E}^{\mathbf{P}}[D_T])^2}\right). \quad (3)$$

We can see

$$\mathbf{E}^{\mathbf{P}}[D_T] = \mathbf{E}^{\mathbf{P}}[F \cdot \mathbf{1}_{\{F \le V_T\}} + V_T \cdot \mathbf{1}_{\{0 \le V_T < F\}}] = F \cdot \mathbf{P}(F \le V_T) + \mathbf{E}^{\mathbf{P}}[V_T \cdot \mathbf{1}_{\{0 \le V_T < F\}}], \quad (4)$$

$$\mathbf{E}^{\mathbf{P}}[(D_T)^2] = \mathbf{E}^{\mathbf{P}}[(F \cdot \mathbf{1}_{\{F \le V_T\}} + V_T \cdot \mathbf{1}_{\{0 \le V_T < F\}})^2] = F^2 \cdot \mathbf{P}(F \le V_T) + \mathbf{E}^{\mathbf{P}}[(V_T)^2 \cdot \mathbf{1}_{\{0 \le V_T < F\}}]. \quad (5)$$

By substituting (4) and (5) into (3), we obtain the equation satisfied by F, provided the initial debt amount D_0 and all the parameters are given.

Although the solvent probability $\mathbf{P}(F \le V_T)$, and the expectations $\mathbf{E}^{\mathbf{P}}[V_T \cdot \mathbf{1}_{\{0 \le V_T < F\}}]$ and $\mathbf{E}^{\mathbf{P}}[(V_T)^2 \cdot \mathbf{1}_{\{0 \le V_T < F\}}]$ can be numerically obtained via Monte Carlo simulation, they can be reduced to calculation of some improper integrals with respect to standard normal distribution as is seen below. Thus we can numerically solve the nonlinear equation for the debt face value F.

Let

$$\Psi = V_0^O e^{-\eta T} + V_0^{NO} e^{rT} + I_S - I_C.$$

In the remainder of this section, we pay attention to the expression of $W_T^C = \rho W_T^S + \sqrt{1-\rho^2}W_T'$, where W_t' is a standard Brownian motion independent of W_t^S. Thus the pair (W_T^S, W_T^C) follows two dimensional normal distribution with variance T and correlation ρ, the pair (W_T^S, W_T^C) has the same distribution as $(\sqrt{T}Z_1, \sqrt{T}(\rho Z_1 + \sqrt{1-\rho^2}Z_2))$ where Z_1 and Z_2 are independent standard normal variables.

Also, let $\phi(z) = \frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}$ be the standard normal density, and $\Phi(z) = \int_{-\infty}^{z} \phi(x)dx$ be the standard normal distribution function.

Proposition 1. *We have*

$$\mathbf{P}(F \le V_T) = \int_{-\infty}^{\infty} \Phi\left(\frac{\log \frac{\Psi + \kappa_S I_S \cdot e^{\sigma_S\sqrt{T}z_1 - \frac{(\sigma_S)^2}{2}T} - F}{\kappa_C I_C} - \sigma_C\sqrt{T}\rho z_1 + \frac{(\sigma_C)^2}{2}T}{\sigma_C\sqrt{T}\sqrt{1-\rho^2}}\right)\phi(z_1)dz_1. \quad (6)$$

Proof. Since (W_T^S, W_T^C) has the same distribution as $(\sqrt{T}Z_1, \sqrt{T}(\rho Z_1 + \sqrt{1-\rho^2}Z_2)$ with some independent standard normal variables Z_1 and Z_2, we have

$$
\begin{aligned}
&\mathbf{P}(F \le V_T) \\
&= \mathbf{P}\left(F \le \Psi + \kappa_S I_S \cdot e^{\sigma_S W_T^S - \frac{(\sigma_S)^2}{2}T} - \kappa_C I_C \cdot e^{\sigma_C W_T^C - \frac{(\sigma_C)^2}{2}T} \right) \\
&= \mathbf{P}\left(F \le \Psi + \kappa_S I_S \cdot e^{\sigma_S \sqrt{T} Z_1 - \frac{(\sigma_S)^2}{2}T} - \kappa_C I_C \cdot e^{\sigma_C \sqrt{T}(\rho Z_1 + \sqrt{1-\rho^2} Z_2) - \frac{(\sigma_C)^2}{2}T} \right) \\
&= \int_{-\infty}^{\infty} \mathbf{P}\left(F \le \Psi + \kappa_S I_S \cdot e^{\sigma_S \sqrt{T} z_1 - \frac{(\sigma_S)^2}{2}T} - \kappa_C I_C \cdot e^{\sigma_C \sqrt{T}(\rho z_1 + \sqrt{1-\rho^2} Z_2) - \frac{(\sigma_C)^2}{2}T} \mid Z_1 = z_1 \right) \phi(z_1) dz_1 \\
&= \int_{-\infty}^{\infty} \mathbf{P}\left(Z_2 \le \frac{\log \frac{\Psi + \kappa_S I_S \cdot e^{\sigma_S \sqrt{T} z_1 - \frac{(\sigma_S)^2}{2}T} - F}{\kappa_C I_C} - \sigma_C \sqrt{T} \rho z_1 + \frac{(\sigma_C)^2}{2}T}{\sigma_C \sqrt{T}\sqrt{1-\rho^2}} \right) \phi(z_1) dz_1 \\
&= \int_{-\infty}^{\infty} \Phi\left(\frac{\log \frac{\Psi + \kappa_S I_S \cdot e^{\sigma_S \sqrt{T} z_1 - \frac{(\sigma_S)^2}{2}T} - F}{\kappa_C I_C} - \sigma_C \sqrt{T} \rho z_1 + \frac{(\sigma_C)^2}{2}T}{\sigma_C \sqrt{T}\sqrt{1-\rho^2}} \right) \phi(z_1) dz_1.
\end{aligned}
$$

□

Here we set for $\xi_1, \xi_2, y \in \mathbb{R}$,

$$
\Xi(z; \xi_1, \xi_2, y) := \frac{\log \frac{\Psi + \kappa_S I_S \cdot e^{\sigma_S \sqrt{T} z_1 + \xi_1 T - y}}{\kappa_C I_C} - \sigma_C \sqrt{T} \rho z_1 + \xi_2 T}{\sigma_C \sqrt{T}\sqrt{1-\rho^2}}. \tag{7}
$$

Then we can express the solvent probability (6) in the form of

$$
\mathbf{P}(F \le V_T) = \int_{-\infty}^{\infty} \Phi\left(\Xi\left(z_1; -\frac{(\sigma_S)^2}{2}, \frac{(\sigma_C)^2}{2}, F \right) \right) \phi(z_1) dz_1.
$$

In order to achieve the formula for $\mathbf{E}^{\mathbf{P}}[D_T]$, we need to calculate $\mathbf{E}^{\mathbf{P}}[V_T \cdot \mathbf{1}_{\{0 \le V_T < F\}}]$. Immedeiately we can see

$$
\begin{aligned}
&\mathbf{E}^{\mathbf{P}}[V_T \cdot \mathbf{1}_{\{0 \le V_T < F\}}] \\
&= \mathbf{E}^{\mathbf{P}}[\left(\Psi + \kappa_S I_S \cdot e^{\sigma_S W_T^S - \frac{(\sigma_S)^2}{2}T} - \kappa_C I_C \cdot e^{\sigma_C W_T^C - \frac{(\sigma_C)^2}{2}T} \right) \cdot \mathbf{1}_{\{0 \le V_T < F\}}] \\
&= \Psi \cdot \mathbf{P}(0 \le V_T < F) + \kappa_S I_S \mathbf{E}^{\mathbf{P}}[e^{\sigma_S W_T^S - \frac{(\sigma_S)^2}{2}T} \mathbf{1}_{\{0 \le V_T < F\}}] - \kappa_C I_C \mathbf{E}^{\mathbf{P}}[e^{\sigma_C W_T^C - \frac{(\sigma_C)^2}{2}T} \mathbf{1}_{\{0 \le V_T < F\}}].
\end{aligned}
$$

As for the insolvent probability $\mathbf{P}(0 \le V_T < F)$, it follows from Proposition 1 with (7) that

$$
\begin{aligned}
&\mathbf{P}(0 \le V_T < F) = \mathbf{P}(0 \le V_T) - \mathbf{P}(F \le V_T) \\
&\quad = \int_{-\infty}^{\infty} \left\{ \Phi\left(\Xi\left(z_1; -\frac{(\sigma_S)^2}{2}, \frac{(\sigma_C)^2}{2}, 0 \right) \right) - \Phi\left(\Xi\left(z_1; -\frac{(\sigma_S)^2}{2}, \frac{(\sigma_C)^2}{2}, F \right) \right) \right\} \phi(z_1) dz_1. \qquad (8)
\end{aligned}
$$

For calculation of the other expectations, we need some preparation. Denote by $\mathbf{P}^{\mathbf{h}}$ another probability measure specified for a pair of $(\mathcal{F}_t)$-adapted processes $\mathbf{h}(t) = (h_1(t), h_2(t))$ that is defined via the Radon-Nikodym density

$$
\frac{d\mathbf{P}^{\mathbf{h}}}{d\mathbf{P}} = \exp\left(\int_0^T h_1(t) W_t^S - \frac{1}{2}\int_0^T h_1(t)^2 dt + \int_0^T h_2(t) W_t' - \frac{1}{2}\int_0^T h_2(t)^2 dt \right).
$$

It follows from Girsanov-Maruyama theorem that

$$
\tilde{W}_t^S := W_t^S - \int_0^t h_1(u) du, \quad \tilde{W}_t' := W_t' - \int_0^t h_2(u) du
$$

are independent standard Brownian motions under the new measure $\mathbf{P}^{\mathbf{h}}$.

Then, we have the following proposition.

Proposition 2. *We have*

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}}[e^{\sigma_S W_T^S - \frac{(\sigma_S)^2}{2}T}\mathbf{1}_{\{0\le V_T<F\}}] = \mathbf{P}^{(\sigma_S,0)}(0\le V_T<F)\\
&\quad = \int_{-\infty}^{\infty}\left\{\Phi\left(\Xi\left(z_1;\frac{(\sigma_S)^2}{2},\frac{(\sigma_C)^2}{2}-\rho\sigma_S\sigma_C,0\right)\right)\right.\\
&\qquad\qquad \left.-\Phi\left(\Xi\left(z_1;\frac{(\sigma_S)^2}{2},\frac{(\sigma_C)^2}{2}-\rho\sigma_S\sigma_C,F\right)\right)\right\}\phi(z_1)dz_1, \qquad (9)\\
&\mathbf{E}^{\mathbf{P}}[e^{\sigma_C W_T^C - \frac{(\sigma_C)^2}{2}T}\mathbf{1}_{\{0\le V_T<F\}}] = \mathbf{P}^{(\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}(0\le V_T<F)\\
&\quad = \int_{-\infty}^{\infty}\left\{\Phi\left(\Xi\left(z_1;-\frac{(\sigma_S)^2}{2}+\rho\sigma_S\sigma_C,-\frac{(\sigma_C)^2}{2},0\right)\right)\right.\\
&\qquad\qquad \left.-\Phi\left(\Xi\left(z_1;-\frac{(\sigma_S)^2}{2}+\rho\sigma_S\sigma_C,-\frac{(\sigma_C)^2}{2},F\right)\right)\right\}\phi(z_1)dz_1. \qquad (10)
\end{aligned}$$

Remark that we use (6), (8), (9), and (10) to obtain the expectation $\mathbf{E}^{\mathbf{P}}[D_T]$ via (4).

Proof. For the expression (9), we can define the new probability measure $\mathbf{P}^{(\sigma_S,0)}$ using the Radon-Nikodym density $\frac{d\mathbf{P}^{(\sigma_S,0)}}{d\mathbf{P}} = e^{\sigma_S W_T^S - \frac{(\sigma_S)^2}{2}T}$. It follows from Girsanov-Maruyama theorem that $\tilde{W}_t^S = W_t^S - \sigma_S t$ and W'_t are independent standard Brownian motions under the new measure $\mathbf{P}^{(\sigma_S,0)}$.

Thus we have

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}}[e^{\sigma_S W_T^S - \frac{(\sigma_S)^2}{2}T}\mathbf{1}_{\{0\le V_T<F\}}] = \mathbf{P}^{(\sigma_S,0)}(0\le V_T<F)\\
&= \mathbf{P}^{(\sigma_S,0)}\left(0\le \Psi + \kappa_S I_S\cdot e^{\sigma_S(\tilde{W}_T^S+\sigma_S T)-\frac{(\sigma_S)^2}{2}T} - \kappa_C I_C\cdot e^{\sigma_C(\rho(\tilde{W}_T^S+\sigma_S T)+\sqrt{1-\rho^2}W'_T)-\frac{(\sigma_C)^2}{2}T} < F\right)\\
&= \mathbf{P}^{(\sigma_S,0)}\left(0\le \Psi + \kappa_S I_S\cdot e^{\sigma_S\sqrt{T}Z_1+\frac{(\sigma_S)^2}{2}T} - \kappa_C I_C\cdot e^{\sigma_C\sqrt{T}(\rho Z_1+\sqrt{1-\rho^2}Z_2)+\left\{\rho\sigma_S\sigma_C-\frac{(\sigma_C)^2}{2}\right\}T} < F\right)\\
&= \int_{-\infty}^{\infty}\left\{\Phi\left(\frac{\log\frac{\Psi+\kappa_S I_S e^{\sigma_S\sqrt{T}z_1+\frac{(\sigma_S)^2}{2}T}}{\kappa_C I_C} - \rho\sigma_C\sqrt{T}z_1 - \left\{\rho\sigma_S\sigma_C-\frac{(\sigma_C)^2}{2}\right\}T}{\sigma_C\sqrt{T}\sqrt{1-\rho^2}}\right)\right.\\
&\qquad \left. -\Phi\left(\frac{\log\frac{\Psi+\kappa_S I_S e^{\sigma_S\sqrt{T}z_1+\frac{(\sigma_S)^2}{2}T}-F}{\kappa_C I_C} - \rho\sigma_C\sqrt{T}z_1 - \left\{\rho\sigma_S\sigma_C-\frac{(\sigma_C)^2}{2}\right\}T}{\sigma_C\sqrt{T}\sqrt{1-\rho^2}}\right)\right\}\phi(z_1)dz_1.
\end{aligned}$$

For the expression (10), we have

$$\begin{aligned}
\mathbf{E}^{\mathbf{P}}[e^{\sigma_C W_T^C - \frac{(\sigma_C)^2}{2}T}\mathbf{1}_{\{0\le V_T<F\}}] &= \mathbf{E}^{\mathbf{P}}[e^{\sigma_C(\rho W_T^S+\sqrt{1-\rho^2}W'_T)-\frac{(\sigma_C)^2}{2}T}\mathbf{1}_{\{0\le V_T<F\}}]\\
&= \mathbf{E}^{\mathbf{P}}[e^{\rho\sigma_C W_T^S - \frac{(\rho\sigma_C)^2}{2}T + \sqrt{1-\rho^2}\sigma_C W'_T - \frac{(1-\rho^2)(\sigma_C)^2}{2}T}\mathbf{1}_{\{0\le V_T<F\}}].
\end{aligned}$$

Similarly, we can define the new probability measure $\mathbf{P}^{(\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}$ using the Radon-Nikodym density

$$\frac{d\mathbf{P}^{(\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}}{d\mathbf{P}} = e^{\rho\sigma_C W_T^S - \frac{(\rho\sigma_C)^2}{2}T + \sqrt{1-\rho^2}\sigma_C W'_T - \frac{(1-\rho^2)(\sigma_C)^2}{2}T}.$$

It follows from Girsanov-Maruyama theorem that $\tilde{W}_t^S = W_t^S - \rho\sigma_C t$ and $\tilde{W}'_t = W'_t - \sqrt{1-\rho^2}\sigma_C t$ are independent standard Brownian motions under the new measure $\mathbf{P}^{(\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}$.

Therefore

$$\mathbf{E}^{\mathbf{P}}[e^{\sigma_C W_T^C - \frac{(\sigma_C)^2}{2}T}\mathbf{1}_{\{0\le V_T<F\}}] = \mathbf{P}^{(\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}(0 \le V_T < F)$$

$$= \mathbf{P}^{(\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}\Bigg(0 \le \Psi + \kappa_S I_S \cdot e^{\sigma_S(\tilde{W}_T^S+\rho\sigma_C T)-\frac{(\sigma_S)^2}{2}T}$$

$$- \kappa_C I_C \cdot e^{\sigma_C(\rho(\tilde{W}_T^S+\rho\sigma_C T)+\sqrt{1-\rho^2}(\tilde{W}_T'+\sqrt{1-\rho^2}\sigma_C T)-\frac{(\sigma_C)^2}{2}T} < F\Bigg)$$

$$= \mathbf{P}^{(\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}\Bigg(0 \le \Psi + \kappa_S I_S \cdot e^{\sigma_S(\sqrt{T}Z_1+\rho\sigma_C T)-\frac{(\sigma_S)^2}{2}T}$$

$$- \kappa_C I_C \cdot e^{\sigma_C(\rho(\sqrt{T}Z_1+\rho\sigma_C T)+\sqrt{1-\rho^2}(\sqrt{T}Z_2+\sqrt{1-\rho^2}\sigma_C T)-\frac{(\sigma_C)^2}{2}T} < F\Bigg)$$

$$= \int_{-\infty}^{\infty}\Bigg\{\Phi\left(\frac{\log\frac{\Psi+\kappa_S I_S e^{\sigma_S\sqrt{T}z_1+(\rho\sigma_S\sigma_C-\frac{(\sigma_S)^2}{2})T}}{\kappa_C I_C} - \rho\sigma_C\sqrt{T}z_1 - \frac{(\sigma_C)^2}{2}T}{\sigma_C\sqrt{T}\sqrt{1-\rho^2}}\right)$$

$$- \Phi\left(\frac{\log\frac{\Psi+\kappa_S I_S e^{\sigma_S\sqrt{T}z_1+(\rho\sigma_S\sigma_C-\frac{(\sigma_S)^2}{2})T}-F}{\kappa_C I_C} - \rho\sigma_C\sqrt{T}z_1 - \frac{(\sigma_C)^2}{2}T}{\sigma_C\sqrt{T}\sqrt{1-\rho^2}}\right)\Bigg\}\phi(z_1)dz_1.$$

□

Finally, to achieve the formula for $\mathbf{E}^{\mathbf{P}}[(D_T)^2]$, we need to calculate $\mathbf{E}^{\mathbf{P}}[(V_T)^2 \cdot \mathbf{1}_{\{0\le V_T<F\}}]$. We can see

$$\mathbf{E}^{\mathbf{P}}[(V_T)^2 \cdot \mathbf{1}_{\{0\le V_T<F\}}] = \mathbf{E}^{\mathbf{P}}[\left(\Psi + \kappa_S I_S \cdot e^{\sigma_S W_T^S - \frac{(\sigma_S)^2}{2}T} - \kappa_C I_C \cdot e^{\sigma_C W_T^C - \frac{(\sigma_C)^2}{2}T}\right)^2 \mathbf{1}_{\{0\le V_T<F\}}]$$

$$= \Psi^2\mathbf{P}(0 \le V_T < F)$$

$$+ 2\Psi\left(\kappa_S I_S \mathbf{E}^{\mathbf{P}}[e^{\sigma_S W_T^S - \frac{(\sigma_S)^2}{2}T} \cdot \mathbf{1}_{\{0\le V_T<F\}}] - \kappa_C I_C \mathbf{E}^{\mathbf{P}}[e^{\sigma_C W_T^C - \frac{(\sigma_C)^2}{2}T} \cdot \mathbf{1}_{\{0\le V_T<F\}}]\right)$$

$$+ (\kappa_S I_S)^2\mathbf{E}^{\mathbf{P}}[e^{2\sigma_S W_T^S - (\sigma_S)^2 T} \cdot \mathbf{1}_{\{0\le V_T<F\}}] + (\kappa_C I_C)^2\mathbf{E}^{\mathbf{P}}[e^{2\sigma_C W_T^C - (\sigma_C)^2 T} \cdot \mathbf{1}_{\{0\le V_T<F\}}]$$

$$- 2\kappa_S I_S \kappa_C I_C \mathbf{E}^{\mathbf{P}}[e^{\sigma_S W_T^S - \frac{(\sigma_S)^2}{2}T} e^{\sigma_C W_T^C - \frac{(\sigma_C)^2}{2}T} \cdot \mathbf{1}_{\{0\le V_T<F\}}]$$

$$= \Psi^2\mathbf{P}(0 \le V_T < F) + 2\Psi\left(\kappa_S I_S \mathbf{P}^{(\sigma_S,0)}(0 \le V_T < F) - \kappa_C I_C \mathbf{P}^{(\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}(0 \le V_T < F)\right)$$

$$+ (\kappa_S I_S)^2\mathbf{E}^{\mathbf{P}}[e^{2\sigma_S W_T^S - (\sigma_S)^2 T} \cdot \mathbf{1}_{\{0\le V_T<F\}}] + (\kappa_C I_C)^2\mathbf{E}^{\mathbf{P}}[e^{2\sigma_C W_T^C - (\sigma_C)^2 T} \cdot \mathbf{1}_{\{0\le V_T<F\}}]$$

$$- 2\kappa_S I_S \kappa_C I_C \mathbf{E}^{\mathbf{P}}[e^{\sigma_S W_T^S - \frac{(\sigma_S)^2}{2}T} e^{\sigma_C W_T^C - \frac{(\sigma_C)^2}{2}T} \cdot \mathbf{1}_{\{0\le V_T<F\}}].$$

Since we have already obtained $\mathbf{P}(0 \le V_T < F), \mathbf{P}^{(\sigma_S,0)}(0 \le V_T < F)$, and $\mathbf{P}^{(\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}(0 \le V_T < F)$, we have to calculate the remaining three expectations.

The results are as follows. The proof is given in Appendix.

Proposition 3. *We have*

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}}[e^{2\sigma_S W_T^S-(\sigma_S)^2T}\cdot \mathbf{1}_{\{0\le V_T<F\}}]=e^{(\sigma_S)^2T}\mathbf{P}^{(2\sigma_S,0)}(0\le V_T<F)\\
&=e^{(\sigma_S)^2T}\int_{-\infty}^{\infty}\left\{\Phi\left(\Xi\left(z_1;\frac{3(\sigma_S)^2}{2},\frac{(\sigma_C)^2}{2}-2\rho\sigma_S\sigma_C,0\right)\right)\right.\\
&\qquad\left.-\Phi\left(\Xi\left(z_1;\frac{3(\sigma_S)^2}{2},\frac{(\sigma_C)^2}{2}-2\rho\sigma_S\sigma_C,F\right)\right)\right\}\phi(z_1)dz_1,\\
&\mathbf{E}^{\mathbf{P}}[e^{2\sigma_C W_T^C-(\sigma_C)^2T}\mathbf{1}_{\{0\le V_T<F\}}]=e^{(\sigma_C)^2T}\mathbf{P}^{(2\rho\sigma_C,2\sqrt{1-\rho^2}\sigma_C)}(0\le V_T<F)\\
&=e^{(\sigma_C)^2T}\int_{-\infty}^{\infty}\left\{\Phi\left(\Xi\left(z_1;-\frac{(\sigma_S)^2}{2}+2\rho\sigma_S\sigma_C,-\frac{3(\sigma_C)^2}{2},0\right)\right)\right.\\
&\qquad\left.-\Phi\left(\Xi\left(z_1;-\frac{(\sigma_S)^2}{2}+2\rho\sigma_S\sigma_C,-\frac{3(\sigma_C)^2}{2},F\right)\right)\right\}\phi(z_1)dz_1,\\
&\mathbf{E}^{\mathbf{P}}[e^{\sigma_S W_T^S-\frac{(\sigma_S)^2}{2}T}e^{\sigma_C W_T^C-\frac{(\sigma_C)^2}{2}T}\cdot \mathbf{1}_{\{0\le V_T<F\}}]=e^{\rho\sigma_S\sigma_C T}\mathbf{P}^{(\sigma_S+\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}(0\le V_T<F)\\
&=e^{\rho\sigma_S\sigma_C T}\int_{-\infty}^{\infty}\left\{\Phi\left(\Xi\left(z_1;\frac{(\sigma_S)^2}{2}+\rho\sigma_S\sigma_C,-\frac{(\sigma_S)^2}{2}-\rho\sigma_S\sigma_C,0\right)\right)\right.\\
&\qquad\left.-\Phi\left(\Xi\left(z_1;\frac{(\sigma_S)^2}{2}+\rho\sigma_S\sigma_C,-\frac{(\sigma_S)^2}{2}-\rho\sigma_S\sigma_C,F\right)\right)\right\}\phi(z_1)dz_1.
\end{aligned}$$

3 Second model with a single noise following Johnson's SU distribution

We can notice that the model introduced in the last section is tractable enough to numerically solve the equation for F. However it seems a little more useful for the debt valuation to calculate $\mathbf{P}(F\le V_T)$, $\mathbf{E}^{\mathbf{P}}[V_T\cdot \mathbf{1}_{\{0\le V_T<F\}}]$ and $\mathbf{E}^{\mathbf{P}}[V_T^2\cdot \mathbf{1}_{\{0\le V_T<F\}}]$ appeared in (4) and (5) with some parametric distribution.

We modeled revenue and operating expenses separately in the previous section. However, for the purpose of evaluating debt value, it is sufficient to focus only on the total profit, which is the difference between them.

From (1) and (2), it follows

$$\begin{aligned}
V_T&=V_0^Oe^{-\eta T}+V_0^{NO}e^{rT}+\left(1+\kappa_S e^{\sigma_S W_T^S-\frac{(\sigma_S)^2}{2}T}\right)I_S-\left(1+\kappa_C e^{\sigma_C W_T^C-\frac{(\sigma_C)^2}{2}T}\right)I_C\\
&=\mathbf{E}^{\mathbf{P}}[V_T]+\kappa_S I_S e^{\sigma_S W_T^S-\frac{(\sigma_S)^2}{2}T}-\kappa_C I_C e^{\sigma_C W_T^C-\frac{(\sigma_C)^2}{2}T}-(\kappa_S I_S-\kappa_C I_C).
\end{aligned}$$

We introduce a parameter σ to specify the variance of V_T as follows.

$$\begin{aligned}
\mathrm{Var}(V_T)&=\mathbf{E}^{\mathbf{P}}\left[\left(\kappa_S I_S e^{\sigma_S W_T^S-\frac{(\sigma_S)^2}{2}T}-\kappa_C I_C e^{\sigma_C W_T^C-\frac{(\sigma_C)^2}{2}T}-(\kappa_S I_S-\kappa_C I_C)\right)^2\right]\\
&=(e^{(\sigma_S)^2T}-1)\kappa_S^2I_S^2+(e^{(\sigma_C)^2T}-1)\kappa_C^2I_C^2-2(e^{\rho\sigma_S\sigma_C T}-1)\kappa_S\kappa_C I_S I_C\\
&=:\sigma^2T. \qquad (11)
\end{aligned}$$

From the above, we introduce our second model, in which the terminal asset value is different from that of the first model and is denoted by $\tilde{V}_T$. We suppose that the terminal asset value $\tilde{V}_T$ is given as follows.

$$\tilde{V}_T=\mathbf{E}^{\mathbf{P}}[V_T]+\sigma\sqrt{T}\varepsilon,$$

where V_T is the terminal asset value of the first model given in (1), and ε is a random variable following some parametric distribution with zero mean and unit variance.

Here we assume that the random variable ε follows Johnson's SU distribution, which has been often used in quantitative finance like modeling asset returns as an alternative to the normal distribution because it has four parameters so that the skewness and kurtosis can be adjusted relatively freely while it is based on the normal distribution.

Specifically, Johnson's SU distribution is a probability distribution with four parameters $\gamma, \delta, \lambda, \xi$ where $\delta > 0, \lambda > 0$. Its distribution function is given by

$$F_{\mathrm{JSU}}(x) = \Phi\left(\gamma + \delta \sinh^{-1}\left(\frac{x-\xi}{\lambda}\right)\right),$$

and its density is given by

$$f_{\mathrm{JSU}}(x) = \frac{\delta}{\lambda\sqrt{2\pi}\sqrt{\left(\frac{x-\xi}{\lambda}\right)^2+1}} \exp\left\{-\frac{1}{2}\left(\gamma + \delta \sinh^{-1}\left(\frac{x-\xi}{\lambda}\right)^2\right)\right\}.$$

It is known that the mean, variance, skewness, and kurtosis of Johnson's SU distribution are respectively achieved as follows (for example, see Naguez and Prigent [7].)

Setting $\mathit{\Delta} = \exp(\delta^{-2})$ and $\mathit{\Gamma} = \dfrac{\gamma}{\delta}$, we have

- Mean: $\xi - \lambda \mathit{\Delta}^{\frac{1}{2}} \sinh(\mathit{\Gamma})$.
- Variance: $\dfrac{\lambda^2}{2}(\mathit{\Delta}-1)(\mathit{\Delta}\cosh(2\mathit{\Gamma})+1)$.
- Skewness:
$$-\operatorname{sgn}(\mathit{\Gamma})\sqrt{\mathit{\Delta}(\mathit{\Delta}-1)\frac{|\mathit{\Delta}(\mathit{\Delta}+2)\sinh(3\mathit{\Gamma})+3\sinh(\mathit{\Gamma})|^2}{2\left(\mathit{\Delta}\cosh(2\mathit{\Gamma})+1\right)^3}}, \tag{12}$$
  where $\operatorname{sgn}(\cdot)$ is the sign function.
- Kurtosis:
$$\frac{\mathit{\Delta}^2(\mathit{\Delta}^4+2\mathit{\Delta}^3+3\mathit{\Delta}^2-3)\cosh(4\mathit{\Gamma})+4\mathit{\Delta}^2(\mathit{\Delta}+2)\cosh(2\mathit{\Gamma})+3(2\mathit{\Delta}+1)}{2\left(\mathit{\Delta}\cosh(2\mathit{\Gamma})+1\right)^2}. \tag{13}$$

We remark that the skewness and kurtosis depend only on the two parameters γ and δ.

In order to estimate the parameters $(\gamma, \delta, \lambda, \xi)$ of Johnson's SU distribution for the random variable ε, we match the skewness and kurtosis of Johnson's SU distribution to those of the terminal asset value V_T given in (1), as skewness and kurtosis are invariant to affine transformation.

Specifically, we first match the skewness and kurtosis between the terminal asset value V_T of the first model and Johnson's SU distribution. To achieve this, we solve the following minimization problem to obtain the estimates $(\hat{\gamma}, \hat{\delta})$ that minimize the objective function to zero:

$$(\hat{\gamma}, \hat{\delta}) = \arg\min\left(\mathrm{Skew}_{\mathrm{JSU}}(\gamma,\delta) - \mathrm{Skew}(V_T)\right)^2 + \left(\mathrm{Kurt}_{\mathrm{JSU}}(\gamma,\delta) - \mathrm{Kurt}(V_T)\right)^2, \tag{14}$$

where $\mathrm{Skew}_{\mathrm{JSU}}(\gamma,\delta)$ and $\mathrm{Kurt}_{\mathrm{JSU}}(\gamma,\delta)$ are given in (12) and (12) respectively, and $\mathrm{Skew}(V_T)$ and $\mathrm{Kurt}(V_T)$ are given in (17) in Appendix B.

Then, since ε is supposed to have zero mean and unit variance, we can obtain the remaining two parameters $(\hat{\xi}, \hat{\lambda})$ by solving equations

$$\xi - \lambda\hat{\mathit{\Delta}}^{\frac{1}{2}}\sinh(\hat{\mathit{\Gamma}}) = 0, \ \frac{\lambda^2}{2}(\hat{\mathit{\Delta}}-1)\left(\hat{\mathit{\Delta}}\cosh(2\hat{\mathit{\Gamma}})+1\right) = 1,$$

where $\hat{\mathit{\Delta}} = \exp(\hat{\delta}^{-2})$ and $\hat{\mathit{\Gamma}} = \dfrac{\hat{\gamma}}{\hat{\delta}}$.

We remark that we need to calculate $\mathbf{E}^{\mathbf{P}}[D_T]$ and $\mathbf{E}^{\mathbf{P}}[(D_T)^2]$ under the second model, more specifically, $\mathbf{P}(F \le \tilde{V}_T)$, $\mathbf{E}^{\mathbf{P}}[\tilde{V}_T \cdot \mathbf{1}_{\{0\le \tilde{V}_T<F\}}]$ and $\mathbf{E}^{\mathbf{P}}[\tilde{V}_T^2 \cdot \mathbf{1}_{\{0\le \tilde{V}_T<F\}}]$ to obtain the equation for F in (3).

Under our second model, we remark that

$$F \le \mathbf{E}^{\mathbf{P}}[V_T] + \sigma\sqrt{T}\varepsilon \iff \theta_1 \le \varepsilon, \quad 0 \le \mathbf{E}^{\mathbf{P}}[V_T] + \sigma\sqrt{T}\varepsilon < F \iff \theta_2 \le \varepsilon < \theta_1,$$

where V_T is given in (1), and we set

$$\theta_1 = \frac{F - \mathbf{E}^{\mathbf{P}}[V_T]}{\sigma\sqrt{T}}, \quad \theta_2 = \frac{-\mathbf{E}^{\mathbf{P}}[V_T]}{\sigma\sqrt{T}}.$$

At last, we have

$$\begin{aligned}
\mathbf{P}(F \le \tilde{V}_T) &= 1 - \mathbf{P}(\varepsilon < \theta_1) = 1 - \hat{F}_{\mathrm{JSU}}(\theta_1), \\
\mathbf{E}^{\mathbf{P}}[\tilde{V}_T \cdot \mathbf{1}_{\{0\le \tilde{V}_T<F\}}] &= \mathbf{E}^{\mathbf{P}}[V_T]\mathbf{P}\left(\theta_2 \le \varepsilon < \theta_1\right) + \sigma\sqrt{T}\mathbf{E}^{\mathbf{P}}[\varepsilon \cdot \mathbf{1}_{\{\theta_2 \le \varepsilon < \theta_1\}}] \\
&= \mathbf{E}^{\mathbf{P}}[V_T]\left(\hat{F}_{\mathrm{JSU}}(\theta_1) - \hat{F}_{\mathrm{JSU}}(\theta_2)\right) + \sigma\sqrt{T}\int_{\theta_2}^{\theta_1} x \cdot \hat{f}_{\mathrm{JSU}}(x)dx,
\end{aligned}$$

where $\hat{F}_{\mathrm{JSU}}(x)$ and $\hat{f}_{\mathrm{JSU}}(x)$ are respectively Johnson's SU distribution function and its density function with the estimated parameters $(\hat{\gamma}, \hat{\delta}, \hat{\xi}, \hat{\lambda})$.

Similarly, for $\mathbf{E}^{\mathbf{P}}[\tilde{V}_T^2 \cdot \mathbf{1}_{\{0\le \tilde{V}_T<F\}}]$, we have

$$\begin{aligned}
&\mathbf{E}^{\mathbf{P}}[\tilde{V}_T^2 \cdot \mathbf{1}_{\{0\le V_T<F\}}] \\
&= \mathbf{E}^{\mathbf{P}}\left[(\mathbf{E}^{\mathbf{P}}[V_T] + \sigma\sqrt{T}\varepsilon)^2 \cdot \mathbf{1}_{\{0\le \mathbf{E}^{\mathbf{P}}[V_T]+\sigma\sqrt{T}\varepsilon<F\}}\right] \\
&= \mathbf{E}^{\mathbf{P}}[V_T]^2\left(\hat{F}_{\mathrm{JSU}}(\theta_1) - \hat{F}_{\mathrm{JSU}}(\theta_2)\right) - 2\mathbf{E}^{\mathbf{P}}[V_T]\sigma\sqrt{T}\int_{\theta_2}^{\theta_1} x \cdot \hat{f}_{\mathrm{JSU}}(x)dx + \sigma^2 T\int_{\theta_2}^{\theta_1} x^2 \cdot \hat{f}_{\mathrm{JSU}}(x)dx,
\end{aligned}$$

where V_T is given in (1).

4 Numerical illustration

This section presents some numerical illustration on the debt valuation for our first and second model. Specifically we numerically solve the nonlinear equation (3) to obtain the debt face value F and the corresponding credit spread $k_D - r$, where k_D is the credit yield of the zero coupon debt given by the relation $D_0 = Fe^{-k_D T}$. For the purpose, we specify the positive increasing functions $s(v)$ and $c(v)$ for the expected instantaneous revenue and expected instantaneous operating expenses.

In general, it is natural to assume that $s(v)$ is concave. This is because, even if an increase in operating assets leads to higher production, the corresponding rise in product prices is likely to be limited due to oversupply, eventually it can cause the growth in sales to slow down. On the other hand, the instantaneous operating expenses $c(v)$ should be modeled as a convex function of operating assets. This is because, while improved productivity through economies of scale can reduce costs, incorporating a penalty for carbon emissions—reflecting climate change risks—suggests that excessive production may lead to disproportionately higher costs.

For our numerical illustration, we assume $s(v) = a_S + b_S \log(v)$ and $c(v) = a_C + b_C V_u^O + \tilde{b}_C (V_u^O)^2$, where $\sigma_S, a_S, b_S, \sigma_C, a_C, b_C$ and $\tilde{b}_C$ are positive parameters[2]. We note that the specification of the logarithmic function for revenue and the quadratic function for operating expenses is inspired by the production function and the cost function related to carbon emissions in Bourgey et al. [2].

[2] The numerical example shown below satisfies the anti-logarithm condition $V_0^O e^{-\eta T} > 1$.

Under the specification of $s(v) = a_S + b_S \log(v)$ and $c(v) = a_C + b_C V_u^O + \tilde{b}_C (V_u^O)^2$, we have

$$I_S = \int_0^t \{a_S + b_S \log V_u^O\} du = (a_S + b_S \log V_0^O)t - \frac{\eta b_S}{2} t^2,$$
$$I_C = \int_0^t \{a_C + b_C V_u^O + \tilde{b}_C (V_u^O)^2\} du = a_C t + b_C V_0^O \frac{1 - e^{-\eta t}}{\eta} + \tilde{b}_C (V_0^O)^2 \frac{1 - e^{-2\eta t}}{2\eta}.$$

To illustrate a numerical example, we assume the following conditions regarding the initial balance sheet of a firm. The firm initially consists solely of 800 million in equity, then raises an additional 500 million by issuing a discount bond with maturity $T = 5$ years and face value F. In short, we consider a firm with total assets amounting to 1.3 billion ($V_0 = 1,300$), funded by 800 million in equity ($E_0 = 800$) and 500 million in debt ($D_0 = 500$). In determining these values, we ensure that the ratios of total assets, debt, and equity are reasonably close to those observed in actual corporate balance sheets. Specifically, we approximate the ratio observed in Japanese manufacturing firms, where debt comprises approximately 40% of the total assets. Similarly, the ratio of non-operating assets to total assets is supposed to be set at approximately 40%, corresponding to the current assets' ratio to total assets in Japanese manufacturing firms. Thus we set $V_0^{NO} = 570$ and $V_0^O = 730$.

We also assume the risk-free interest rate $r = 0.01$ and the operating-asset depreciation rate $\eta = 0.03$.

Next, the parameter values for the logarithmic function for revenue and the quadratic function for operating expenses as introduced above are also set with reference to the income statements of Japanese manufacturing firms, in consideration of the ratios of revenue, profit, and expenses. Specifically, the parameter values of the functions $s(v)$ and $c(v)$ for instantaneous revenue and operational expenses are chosen so that the annual revenue level is around 80% of the total assets, and expenses are approximately 90% of the annual revenue, so we finally set these functions as

$$s(v) = 126 \log(v), \quad c(v) = 73 + V_u^O + 0.0001 (V_u^O)^2. \tag{15}$$

Third, we set the parameters for uncertainty of revenue and expenses of the firm as follows: $\kappa_S = \kappa_C = 0.25, \sigma_S = 0.1, \sigma_C = 0.2$, and $\rho = 0.2$.

Lastly we set the market price of debt risk $p = 0.02$.

Table 1 summarizes the basic setting values of the model parameters other than those contained in the functions $s(v)$ and $c(v)$.

Table 1: The basic setting values of the model parameters other than those contained in the functions $s(v)$ and $c(v)$.

T	D_0	E_0	V_0^{NO}	V_0^O	r	η	κ_S	κ_C	σ_S	σ_C	ρ	p
5	500	800	570	730	0.01	0.03	0.25	0.25	0.1	0.2	0.2	0.02

These parameters are set arbitrarily for the numerical experiments, but we believe that they are not particularly unnatural[3].

4.1 Illustrations of the first model

Under these parameters, Table 2 presents the results for the first model with the parameters given in Table 1 with the functions $s(v)$ and $c(v)$ supposed in (15), where solve the nonlinear equation (3) to determine the debt face value F and the corresponding credit spread k_D as the market price of debt risk p varies from 0.01 to 0.1. For comparison, we include results obtained using the numerical integration method discussed in Section 2, as well as those derived from a Monte Carlo simulation with one million runs.

[3] Indeed, the total asset amount, revenue, and operational expenses are determined with reference to past financial statements of DENSO CORPORATION.

We observe that both the debt face value and the corresponding credit spread get larger as the market price of debt risk p increases. Moreover, we can see that the credit spreads derived range from 0.6% to 1.0% and they are consistent with the actual credit spreads (for 5-year bonds) ranging from 0.29% to 1.25% for AAA to BBB-rated corporate bonds as of November 1, 2024.

Table 2: The debt face value F of for the first model obtained by solving the equation (3) and the corresponding credit spread k_D as the market price of debt risk p varied from 0.01 to 0.1. The top two rows represent the results obtained via numerical integration, while the bottom two rows shows the results obtained through Monte Carlo simulation.

	p	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
N. I.	F	542.46	543.32	544.17	545.03	545.90	546.77	547.64	548.51	549.39	550.27
	k_D	0.630%	0.662%	0.693%	0.725%	0.756%	0.788%	0.820%	0.852%	0.884%	0.916%
M. C.	F	542.67	543.53	544.40	545.27	545.14	546.02	547.90	548.78	549.57	550.56
	k_D	0.637%	0.670%	0.701%	0.733%	0.765%	0.797%	0.830%	0.862%	0.894%	0.926%

Next, we fix the the market price of debt risk $p = 0.02$ to obtain the debt face value F and the corresponding credit spread k_D as the correlation parameter ρ varies from -1 to 1.

Figure 1 shows the relation between the correlation ρ and the debt face value F (left panel) and the corresponding credit spread k_D (right panel) for each pair of the parameters $(\sigma_S, \sigma_C) = (0.1, 0.1), (0.2, 0.1), (0.1, 0.2), (0.2, 0.2)$.

As we can see, the higher the correlation ρ, the lower both the debt face value F and the corresponding credit spread k_D. Additionally, the credit spread tends to be higher as the firm's profit becomes more volatile; in other words, as the parameters (σ_S, σ_C) increase. Furthermore, the uncertainty of operating expenses, σ_C, has a greater impact on the level of the credit spread than the uncertainty of revenue, σ_S. When $\sigma_C > \sigma_S$, the spread remains positive even as the correlation approaches one, whereas in other cases, the spread converges to zero as ρ approaches one.

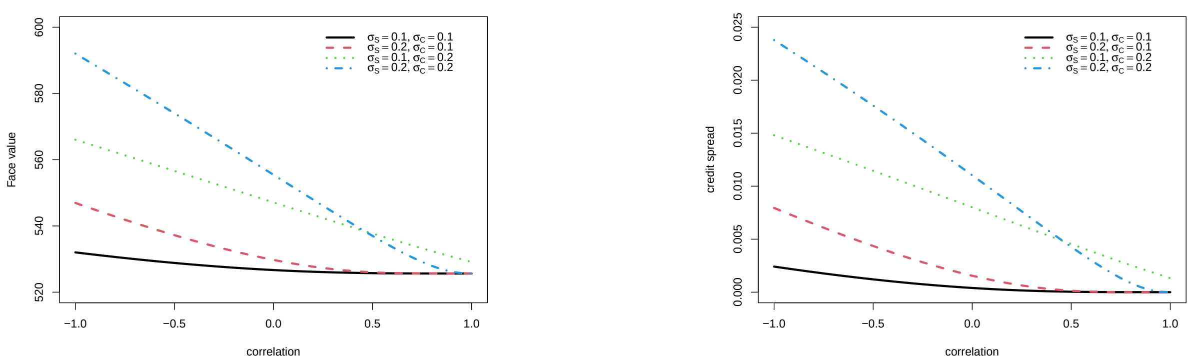

Figure 1: The relation between the correlation ρ and the debt face value F (left panel) and the corresponding credit spread k_D (right panel) for each pair $(\sigma_S, \sigma_C) = (0.1, 0.1), (0.2, 0.1), (0.1, 0.2), (0.2, 0.2)$. The results obtained via numerical integration.

Then Figure 2 shows the term structure of credit spread k_D for each correlation parameter $\rho = -0.2, 0, 0.2, 0.5$.

The credit spread term structure curve resembles that of the classical Merton model, in that we observe the spread initially rising sharply with the length of the term-to-maturity, but then gradually beginning to decrease at a certain point.

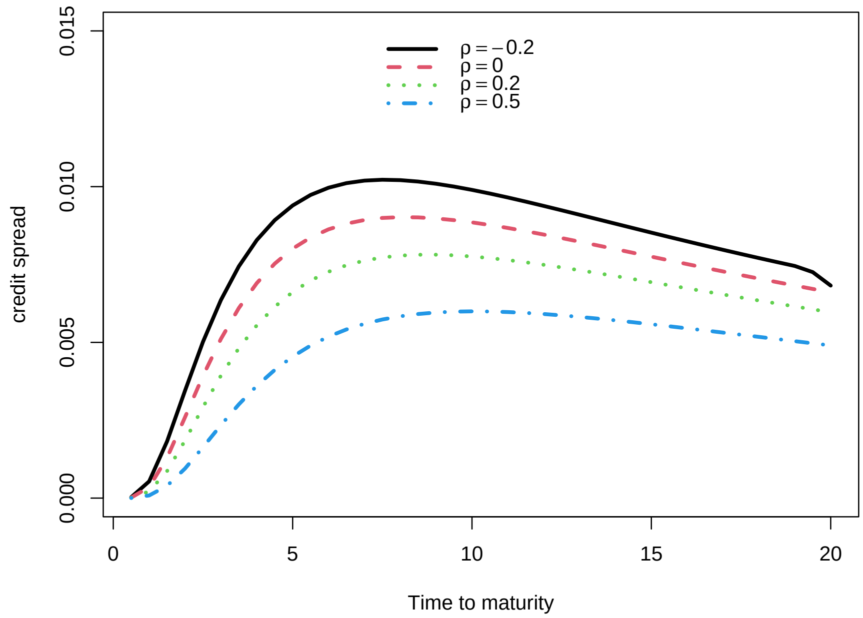

Figure 2: The term structure of credit spread up to maturity of 20 years for each correlation $\rho = -0.2, 0, 0.2, -0.5$. The results obtained via numerical integration.

4.2 Illustrations of the second model

As for the second model with Johnson's SU distribution, we need to estimate the four parameters $(\gamma, \delta, \eta, \xi)$. Under the above parameters, the theoretical skewness and kurtosis of the terminal total asset value V_T are respectively -1.12836 and 6.26984. Solving the minimization problem for matching the third and fourth moments given in (14) implies $\hat{\gamma} = 1.44495$ and $\hat{\delta} = 2.01810$, and then we have

$$\hat{\eta} = \sqrt{\frac{2}{(\hat{\Delta}-1)\left(\hat{\Delta}\cosh(2\hat{\Gamma})+1\right)}} = 1.37000, \quad \hat{\xi} = \hat{\eta}\hat{\Delta}^{\frac{1}{2}}\sinh(\hat{\Gamma}) = 1.20626,$$

where $\hat{\Delta} = \exp(\hat{\delta}^{-2})$ and $\hat{\Gamma} = \frac{\hat{\gamma}}{\hat{\delta}}$.

Indeed, with the estimated parameters $(\hat{\gamma}, \hat{\delta})$, both the theoretical skewness and kurtosis of Johnson's SU distribution coincide with those of the terminal total asset value V_T.

Table 3 displays the the debt face value F and the corresponding credit spread k_D as the market price of debt risk p varies from 0.01 to 0.1 under the second model with Johnson's SU distribution and the above estimated parameters $(\hat{\gamma}, \hat{\delta}, \hat{\eta}, \hat{\xi})$. In addition, Figure 3 shows the term structure of credit spread k_D for each correlation parameter $\rho = -0.2, 0, 0.2, 0.5$.

We remark that the second model with Johnson's SU distribution gives the higher face value and credit spread in comparison with the result of the first model as seen in Table 2 and Figure 2.

These consequences imply that it is hard to naively see that the distribution of the terminal total asset value V_T can be approximated with the Johnson's SU distribution even if the first four moments can be matched[4]. In reality, the second model is assumed to be used independently of the first model, and the parameters are estimated separately from actual data of revenue and operating expenses. So we believe that the second model is useful in terms of conservative credit risk assessment.

Table 3: The debt face value F of for the second model with Johnson's SU distribution and the above estimated parameters $(\hat{\gamma}, \hat{\delta}, \hat{\eta}, \hat{\xi})$ obtained by solving the equation (3) and the corresponding credit spread k_D as the market price of debt risk p varied from 0.01 to 0.1.

p	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09	0.1
F	548.41	553.88	559.44	565.10	570.84	576.70	582.64	588.70	594.85	601.12
k_D	0.848%	1.047%	1.247%	1.448%	1.650%	1.854%	2.059%	2.266%	2.474%	2.684%

Remark 4. *If we suppose that, under the specified functions $s(v)$ and $c(v)$ as above, the firm aims to maximize the expected profit during the period $[0, T]$, we can obtain the initial amount of such optimal operating assets before considering how to finance the assets. Denote by V_0^{O*} the initial amount of operating assets maximizing $\mathbf{E}^{\mathbf{P}}[S_T - C_T]$.*

If $\tilde{b}_C > 0$, then the optimal V_0^{O*} satisfies the following first order condition

$$\frac{d\mathbf{E}^{\mathbf{P}}[S_T - C_T]}{dV_0^O} = (1+\kappa_S)\frac{b_S T}{V_0^O} - (1+\kappa_C)\left\{b_C\frac{1-e^{-\eta T}}{\eta} - \tilde{b}_C V_0^O \frac{1-e^{-2\eta T}}{\eta}\right\} = 0,$$

[4]The estimated Johnson's SU distribution implies a fatter tail on the negative side compared to the empirical distribution generated from random samples of V_T obtained through Monte Carlo simulation.

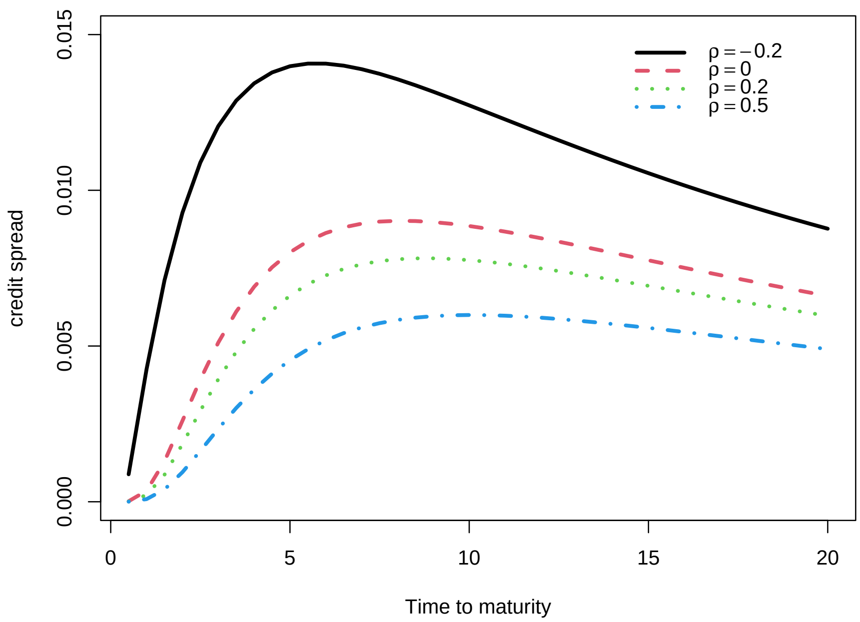

Figure 3: The term structure of credit spread derived by the second model with Johnson's SU distribution and the above estimated parameters, up to maturity of 20 years for each correlation $\rho = -0.2, 0, 0.2, -0.5$.

so the positive solution is given by

$$V_0^{O*} = \frac{-b_C(1-e^{-\eta T}) + \sqrt{\{b_C(1-e^{-\eta T})\}^2 + 4\cdot\dfrac{1+\kappa_S}{1+\kappa_C}\cdot\tilde{b}_C b_S \eta T(1-e^{-2\eta T})}}{2\tilde{b}_C(1-e^{-2\eta T})}.$$

If $\tilde{b}_C = 0$, *we have* $V_0^{O*} = \dfrac{1+\kappa_S}{1+\kappa_C}\cdot\dfrac{b_S\eta T}{b_C(1-e^{-\eta T})}$.

For the parameters given in Table reftable:parameters with the functions $s(v)$ *and* $c(v)$ *supposed in* (15)*, the optimal initial amount of operating assets becomes 132.4232, smaller than 570, which we assume as* V_0^O*.*

Naturally, given the complexity of real-world constraints, the result of this optimization alone is not sufficient to determine the optimal level of operational assets. However, understanding the optimal level of operational assets is useful for determining the conditions for investing in discount bonds. For example, it can be helpful when banks provide financial consulting to borrowing firms.

5 Concluding remarks

In this paper, we consider a firm that begins its business by raising funds through equity and discount bonds at the initial time point. We introduce the model in which the firm's average temporal revenue and operating expenses are formalized as integrals of an increasing function of depreciating operational assets. In addition, we assume that the actual rates of change in revenue and operating expenses fluctuate stochastically according to exponential martingales driven by correlated Brownian motions. Consequently, the terminal asset value of the firm, which includes the total profits up to the bond's maturity, is also subject to uncertainty. Based on this setup, we discuss the appropriate face value of the discount bond and the corresponding credit spread above the risk-free rate, and we can calculate these values numerically under the real probability measure within the Merton model framework.

The results obtained do not significantly differ from those in the classical Merton model, which assumes the firm's asset value follows a one-dimensional geometric Brownian motion. However, introducing a model that captures the uncertainty in revenue and operating expenses—financial flow elements—through two correlated Brownian motions offers promising potential for developing credit risk assessment models using Point of Sale (POS) data or bank transaction data in the future.

As the second model, we also introduce an approach that assumes the uncertainty in the firm's total profit follows Johnson's SU distribution. We estimate the parameters of Johnson's SU distribution to match the mean, variance, skewness, and kurtosis of the first model, and then calculate the appropriate face value and credit spread for the discount bond under this second model. The results indicate a tendency to estimate higher credit risk than the first model. In practice, it is possible to apply the second model based on Johnson's SU distribution independently of the first model. Thus, this approach could be a viable option for conservative credit risk management.

In the future, we aim to explore practical modeling of revenue and operating expenses consistent with actual transaction data, as well as extensions such as debt valuation under incomplete information.

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A Proof of Proposition 3

Similar to the proof of Proposition 2, we can apply equivalent probability measure change and Girsanov-Maruyama theorem to each calculation.

First, we have

$$
\begin{aligned}
&\mathbf{E}^{\mathbf{P}}\big[e^{2\sigma_S W_T^S-(\sigma_S)^2T}\cdot \mathbf{1}_{\{0\le V_T<F\}}\big] = e^{(\sigma_S)^2T}\mathbf{E}^{\mathbf{P}}\big[e^{2\sigma_S W_T^S-\frac{(2\sigma_S)^2}{2}T}\cdot \mathbf{1}_{\{0\le V_T<F\}}\big]\\
&= e^{(\sigma_S)^2T}\mathbf{P}^{(2\sigma_S,0)}(0\le V_T<F)\\
&= e^{(\sigma_S)^2T}\mathbf{P}^{(2\sigma_S,0)}\left(0\le \Psi+\kappa_S I_S\cdot e^{\sigma_S(\tilde{W}_T^S+2\sigma_S T)-\frac{(\sigma_S)^2}{2}T}-\kappa_C I_C\cdot e^{\sigma_C(\rho(\tilde{W}_T^S+2\sigma_S T)+\sqrt{1-\rho^2}W_T')-\frac{(\sigma_C)^2}{2}T}<F\right)\\
&= e^{(\sigma_S)^2T}\mathbf{P}^{(2\sigma_S,0)}\left(0\le \Psi+\kappa_S I_S\cdot e^{\sigma_S\sqrt{T}Z_1+\frac{3(\sigma_S)^2}{2}T}-\kappa_C I_C\cdot e^{\sigma_C\sqrt{T}(\rho Z_1+\sqrt{1-\rho^2}Z_2)+\left\{2\rho\sigma_S\sigma_C-\frac{(\sigma_C)^2}{2}\right\}T}<F\right)\\
&= e^{(\sigma_S)^2T}\int_{-\infty}^{\infty}\left\{\Phi\left(\frac{\log\frac{\Psi+\kappa_S I_S e^{\sigma_S\sqrt{T}z_1+\frac{3(\sigma_S)^2}{2}T}}{\kappa_C I_C}-\rho\sigma_C\sqrt{T}z_1-\left\{2\rho\sigma_S\sigma_C-\frac{(\sigma_C)^2}{2}\right\}T}{\sigma_C\sqrt{T}\sqrt{1-\rho^2}}\right)\right.\\
&\qquad\left.-\Phi\left(\frac{\log\frac{\Psi+\kappa_S I_S e^{\sigma_S\sqrt{T}z_1+\frac{3(\sigma_S)^2}{2}T}-F}{\kappa_C I_C}-\rho\sigma_C\sqrt{T}z_1-\left\{2\rho\sigma_S\sigma_C-\frac{(\sigma_C)^2}{2}\right\}T}{\sigma_C\sqrt{T}\sqrt{1-\rho^2}}\right)\right\}\phi(z_1)dz_1.
\end{aligned}
$$

Next,

$$
\begin{aligned}
&\mathbf{E}^{\mathbf{P}}\big[e^{2\sigma_C W_T^C-(\sigma_C)^2T}\mathbf{1}_{\{0\le V_T<F\}}\big] = \mathbf{E}^{\mathbf{P}}\big[e^{2\sigma_C(\rho W_T^S+\sqrt{1-\rho^2}W_T')-(\sigma_C)^2T}\mathbf{1}_{\{0\le V_T<F\}}\big]\\
&= e^{(\sigma_C)^2T}\mathbf{E}^{\mathbf{P}}\big[e^{2\rho\sigma_C W_T^S-\frac{(2\rho\sigma_C)^2}{2}T+2\sqrt{1-\rho^2}\sigma_C W_T'-\frac{(1-\rho^2)(2\sigma_C)^2}{2}T}\mathbf{1}_{\{0\le V_T<F\}}\big]\\
&= e^{(\sigma_C)^2T}\mathbf{P}^{(2\rho\sigma_C,2\sqrt{1-\rho^2}\sigma_C)}(0\le V_T<F)\\
&= e^{(\sigma_C)^2T}\mathbf{P}^{(2\rho\sigma_C,2\sqrt{1-\rho^2}\sigma_C)}\Bigg(0\le \Psi+\kappa_S I_S\cdot e^{\sigma_S(\tilde{W}_T^S+2\rho\sigma_C T)-\frac{(\sigma_S)^2}{2}T}\\
&\qquad\qquad -\kappa_C I_C\cdot e^{\sigma_C(\rho(\tilde{W}_T^S+2\rho\sigma_C T)+\sqrt{1-\rho^2}(\tilde{W}_T'+2\sqrt{1-\rho^2}\sigma_C T)-\frac{(\sigma_C)^2}{2}T}<F\Bigg)\\
&= e^{(\sigma_C)^2T}\mathbf{P}^{(2\rho\sigma_C,2\sqrt{1-\rho^2}\sigma_C)}\Bigg(0\le \Psi+\kappa_S I_S\cdot e^{\sigma_S\sqrt{T}Z_1+(2\rho\sigma_S\sigma_C-\frac{(\sigma_S)^2}{2})T}\\
&\qquad\qquad -\kappa_C I_C\cdot e^{\sigma_C\sqrt{T}(\rho Z_1+\sqrt{1-\rho^2}Z_2)+\frac{3(\sigma_C)^2}{2}T}<F\Bigg)\\
&= e^{(\sigma_C)^2T}\int_{-\infty}^{\infty}\left\{\Phi\left(\frac{\log\frac{\Psi+\kappa_S I_S e^{\sigma_S\sqrt{T}z_1+(2\rho\sigma_S\sigma_C-\frac{(\sigma_S)^2}{2})T}}{\kappa_C I_C}-\rho\sigma_C\sqrt{T}z_1-\frac{3(\sigma_C)^2}{2}T}{\sigma_C\sqrt{T}\sqrt{1-\rho^2}}\right)\right.\\
&\qquad\left.-\Phi\left(\frac{\log\frac{\Psi+\kappa_S I_S e^{\sigma_S\sqrt{T}z_1+(2\rho\sigma_S\sigma_C-\frac{(\sigma_S)^2}{2})T}-F}{\kappa_C I_C}-\rho\sigma_C\sqrt{T}z_1-\frac{3(\sigma_C)^2}{2}T}{\sigma_C\sqrt{T}\sqrt{1-\rho^2}}\right)\right\}\phi(z_1)dz_1.
\end{aligned}
$$

Finally,

$$
\begin{aligned}
&\mathbf{E}^{\mathbf{P}}\big[e^{\sigma_S W_T^S-\frac{(\sigma_S)^2}{2}T}e^{\sigma_C W_T^C-\frac{(\sigma_C)^2}{2}T}\cdot\mathbf{1}_{\{0\le V_T<F\}}\big]\\
&=\mathbf{E}^{\mathbf{P}}\big[e^{\sigma_S W_T^S-\frac{(\sigma_S)^2}{2}T}e^{\sigma_C(\rho W_T^S+\sqrt{1-\rho^2}W_T')-\frac{(\sigma_C)^2}{2}T}\cdot\mathbf{1}_{\{0\le V_T<F\}}\big]\\
&=e^{\rho\sigma_S\sigma_C T}\mathbf{E}^{\mathbf{P}}\big[e^{(\sigma_S+\rho\sigma_C)W_T^S-\frac{(\sigma_S+\rho\sigma_C)^2}{2}T+\sqrt{1-\rho^2}\sigma_C W_T'-\frac{(1-\rho^2)(\sigma_C)^2}{2}T}\cdot\mathbf{1}_{\{0\le V_T<F\}}\big]\\
&=e^{\rho\sigma_S\sigma_C T}\mathbf{P}^{(\sigma_S+\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}(0\le V_T<F)\\
&=e^{\rho\sigma_S\sigma_C T}\mathbf{P}^{(\sigma_S+\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}\Bigg(0\le\Psi+\kappa_S I_S\cdot e^{\sigma_S(\tilde{W}_T^S+(\sigma_S+\rho\sigma_C)T)-\frac{(\sigma_S)^2}{2}T}\\
&\qquad\qquad-\kappa_C I_C\cdot e^{\sigma_C(\rho(\tilde{W}_T^S+(\sigma_S+\rho\sigma_C)T)+\sqrt{1-\rho^2}(\tilde{W}_T'+\sqrt{1-\rho^2}\sigma_C T)-\frac{(\sigma_C)^2}{2}T}<F\Bigg)\\
&=e^{\rho\sigma_S\sigma_C T}\mathbf{P}^{(\sigma_S+\rho\sigma_C,\sqrt{1-\rho^2}\sigma_C)}\Bigg(0\le\Psi+\kappa_S I_S\cdot e^{\sigma_S\sqrt{T}Z_1+\left\{\frac{(\sigma_S)^2}{2}+\rho\sigma_S\sigma_C\right\}T}\\
&\qquad\qquad-\kappa_C I_C\cdot e^{\sigma_C\sqrt{T}(\rho Z_1+\sqrt{1-\rho^2}Z_2)+\left\{\frac{(\sigma_C)^2}{2}+\rho\sigma_S\sigma_C\right\}T}\le F\Bigg)\\
&=e^{\rho\sigma_S\sigma_C T}\int_{-\infty}^{\infty}\left\{\Phi\left(\frac{\log\frac{\Psi+\kappa_S I_S e^{\sigma_S\sqrt{T}z_1+(\frac{(\sigma_S)^2}{2}+\rho\sigma_S\sigma_C)T}}{\kappa_C I_C}-\rho\sigma_C\sqrt{T}z_1-\left\{\frac{(\sigma_C)^2}{2}+\rho\sigma_S\sigma_C\right\}T}{\sigma_C\sqrt{T}\sqrt{1-\rho^2}}\right)\right.\\
&\qquad\left.-\Phi\left(\frac{\log\frac{\Psi+\kappa_S I_S e^{\sigma_S\sqrt{T}z_1+(\frac{(\sigma_S)^2}{2}+\rho\sigma_S\sigma_C)T}-F}{\kappa_C I_C}-\rho\sigma_C\sqrt{T}z_1-\left\{\frac{(\sigma_C)^2}{2}+\rho\sigma_S\sigma_C\right\}T}{\sigma_C\sqrt{T}\sqrt{1-\rho^2}}\right)\right\}\phi(z_1)dz_1.
\end{aligned}
$$

B Skewness and kurtosis of V_T

We remark that

$$V_T-\mathbf{E}^{\mathbf{P}}[V_T]=\kappa_S I_S e^{\sigma_S W_T^S-\frac{(\sigma_S)^2}{2}T}-\kappa_C I_C e^{\sigma_C W_T^C-\frac{(\sigma_C)^2}{2}T}-(\kappa_S I_S-\kappa_C I_C).$$

Thus the third and fourth centered moments of V_T are respectively obtained as follows.

We have

$$
\begin{aligned}
&\mathbf{E}^{\mathbf{P}}[(V_T - \mathbf{E}^{\mathbf{P}}[V_T])^3] \\
&= (\kappa_S I_S)^3 \left\{ e^{3(\sigma_S)^2 T} - 3e^{(\sigma_S)^2 T} + 2 \right\} - (\kappa_C I_C)^3 \left\{ e^{3(\sigma_C)^2 T} - 3e^{(\sigma_C)^2 T} + 2 \right\} \\
&\quad - (\kappa_S I_S)^2 \kappa_C I_C \left\{ 3e^{\{(\sigma_S)^2 + 2\rho\sigma_S\sigma_C\}T} - 3e^{(\sigma_S)^2 T} - 6e^{\rho\sigma_S\sigma_C T} + 6 \right\} \\
&\quad + \kappa_S I_S (\kappa_C I_C)^2 \left\{ 3e^{\{(\sigma_C)^2 + 2\rho\sigma_S\sigma_C\}T} - 3e^{(\sigma_C)^2 T} - 6e^{\rho\sigma_S\sigma_C T} + 6 \right\}, \\
&\mathbf{E}^{\mathbf{P}}[(V_T - \mathbf{E}^{\mathbf{P}}[V_T])^4] \qquad (16) \\
&= (\kappa_S I_S)^4 \left\{ e^{6(\sigma_S)^2 T} - 4e^{3(\sigma_S)^2 T} + 6e^{(\sigma_C)^2 T} - 3 \right\} - (\kappa_C I_C)^4 \left\{ e^{6(\sigma_C)^2 T} - 4e^{3(\sigma_C)^2 T} + 6e^{(\sigma_S)^2 T} - 3 \right\} \\
&\quad + (\kappa_S I_S)^3 \kappa_C I_C \left\{ -4e^{\{3(\sigma_S)^2 + 3\rho\sigma_S\sigma_C\}T} + 4e^{3(\sigma_S)^2 T} + 12e^{\{(\sigma_S)^2 + 2\rho\sigma_S\sigma_C\}T} - 12e^{(\sigma_S)^2 T} - 12e^{\rho\sigma_S\sigma_C T} + 12 \right\} \\
&\quad + \kappa_S I_S (\kappa_C I_C)^3 \left\{ -4e^{\{3(\sigma_C)^2 + 3\rho\sigma_S\sigma_C\}T} + 4e^{3(\sigma_C)^2 T} + 12e^{\{(\sigma_C)^2 + 2\rho\sigma_S\sigma_C\}T} - 12e^{(\sigma_C)^2 T} - 12e^{\rho\sigma_S\sigma_C T} + 12 \right\} \\
&\quad + (\kappa_S I_S)^2 (\kappa_C I_C)^2 \Big\{ 6e^{\{(\sigma_S)^2 + (\sigma_C)^2 + 4\rho\sigma_S\sigma_C\}T} - 12e^{\{(\sigma_S)^2 + 2\rho\sigma_S\sigma_C\}T} - 12e^{\{(\sigma_C)^2 + 2\rho\sigma_S\sigma_C\}T} \\
&\qquad\qquad + 6e^{(\sigma_S)^2 T} + 6e^{(\sigma_C)^2 T} + 24e^{\rho\sigma_S\sigma_C T} - 18 \Big\}.
\end{aligned}
$$

At last we obtain the skewness and kurtosis of V_T respectively by

$$
\mathrm{Skew}(V_T) = \frac{\mathbf{E}^{\mathbf{P}}[(V_T - \mathbf{E}^{\mathbf{P}}[V_T])^3]}{(\sigma^2 T)^{\frac{3}{2}}}, \ \mathrm{Kurt}(V_T) = \frac{\mathbf{E}^{\mathbf{P}}[(V_T - \mathbf{E}^{\mathbf{P}}[V_T])^4]}{(\sigma^2 T)^2}, \tag{17}
$$

where $\sigma^2 T$, the variance of V_T, is given in (11).