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A Valuation Model and Trading Strategy for U.S. Agency MBSs

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Abstract

A mortgage-backed security (MBS) is a securitized instrument whose cash flows consist of principal and interest payments from an underlying pool of residential mortgage loans. Compared to government and corporate bonds, MBS exhibits more complex cash flow structures due to embedded prepayment options. This paper examines the TBA (To-Be-Announced) market, a highly liquid forward market for U.S. agency MBSs, and investigates whether persistent mispricing exists from which investors can benefit. To this end, we construct a trading strategy based on an arbitrage-free model. We adopt a no-arbitrage pricing framework as the MBS valuation model, incorporating four latent factors that jointly capture interest rate risk, credit risk, and prepayment risk. The model is estimated using observable data such as TBA forward prices. Empirical analysis reveals that the price difference between observed TBA market prices and theoretical prices exhibits mean-reverting behavior over time. Furthermore, the price difference exhibits an inverse smile shape with respect to moneyness, defined as the fixed mortgage rate of each MBS minus the current mortgage rate. TBA contracts near the at-the-money level tend to be overvalued, whereas those further away tend to be undervalued. An arbitrage trading strategy that uses this price discrepancy as a mispricing signal earns an average annualized excess return of 58.27 bps relative to an equally weighted benchmark portfolio. The time-series behavior of excess returns reveals distinct periods of strong outperformance and periods of limited gains.

Keyword: Agency MBSs, TBA market, Arbitrage-free pricing, Trading strategy, Prepayment risk, Fannie Mae

1 Introduction

1.1 Agency MBSs

MBS (mortgage-backed security) is a securitized instrument whose cash flows consist of principal and interest payments from an underlying pool of residential mortgage loans. In the United States, MBSs are issued by entities such as Ginnie Mae (Government National Mortgage Association, GNMA), Fannie Mae (Federal National Mortgage Association, FNMA), and other issuers. Ginnie Mae and Fannie Mae are government-sponsored enterprises (GSEs), and MBS issued by GSEs are referred to as Agency MBSs. Figure 1 illustrates the evolution of outstanding issuance by bond type in the United States. From Figure 1, it can be seen that in 2021, the Agency MBSs market accounted for approximately 20% of the outstanding balance of the U.S. bond market, indicating its significant share in the U.S. bond market. This paper aims to explore whether persistent mispricing exists from which investors can benefit in the U.S. Agency MBSs market by constructing a fixed income trading strategy using an arbitrage-free pricing model.

The novelty of this paper lies in applying an arbitrage-free pricing model to the trading strategy of Agency MBSs and, in particular, targeting the TBA market as the trading object. The advantages of

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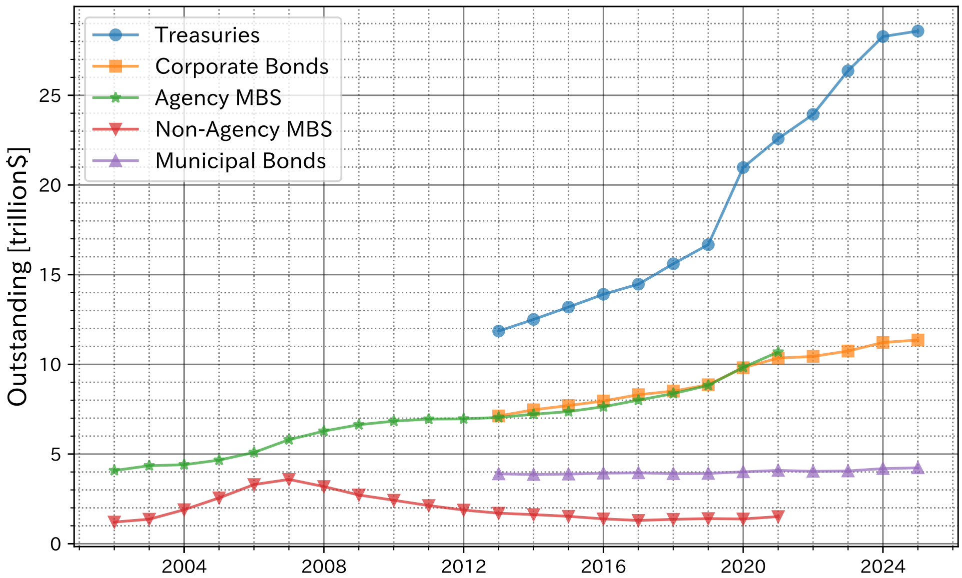

Figure 1: Evolution in outstanding issuance by bond type in the United States. The circular marker (blue), square marker (orange), star marker (green), downward triangle marker (red), and upward triangle marker (purple) represent the issuance of U.S. Treasury bonds, U.S. corporate bonds, U.S. Agency MBSs (Ginnie Mae, Fannie Mae, Freddie Mac), U.S. Non-Agency MBSs (total of CMBSs and RMBSs), and U.S. municipal bonds, respectively. The unit of the vertical axis is trillion$. The data is obtained from the Securities Industry and Financial Markets Association (SIFMA) website (https://www.sifma.org/resources/archive/research/statistics/). Data for Treasury bonds, corporate bonds, and municipal bonds before 2012, as well as for Agency MBSs and Non-Agency MBSs after 2022, is missing.

using an arbitrage-free pricing model are twofold. The first advantage is that there is no need to ensure consistency between the prepayment rates under the physical measure and the risk-neutral measure. In an arbitrage-free pricing model, the latent variable processes included in the model are estimated using the observed MBS prices in the market, so the prepayment rates estimated by the model are those under the risk-neutral measure, which do not necessarily have to match the prepayment rates under the physical measure. The second advantage is that using an arbitrage-free pricing model allows the construction of a consistent pricing model for the TBA market. The TBA market, a liquid market for Agency MBSs, employs forward contracts rather than the MBSs themselves. Therefore, to estimate the model using the prices quoted in the TBA market, a theoretical forward price of MBS as the underlying asset is required. The theoretical forward price of MBS in the arbitrage-free pricing model is derived by B. Dunn, Kargar, and Zheng (2021). Arbitrage-free pricing models for Agency MBSs have been studied in works such as Chernov, B. R. Dunn, and Longstaff (2017). Furthermore, with respect to bond trading strategies using arbitrage-free pricing models, Duarte, Longstaff, and Yu (2006) and Turan Bali and Wu (2009) have constructed trading strategies related to government bonds and swaps using dynamic interest rate term structure models. However, to our knowledge, there are no prior studies discussing bond trading strategies related to Agency MBSs using arbitrage-free pricing models.

MBSs have more complex cash flows compared to government or corporate bonds due to early repayment behavior by borrowers, referred to as prepayment. For reviews on MBSs, see Kau and Keenan (1995), Capone (2001), Hayre (2001), Wallace (2005), and Fabozzi (2016). The cash flows of fixed-coupon MBSs, which are the focus of this paper, consist of monthly fixed payments of principal and interest made by the borrowers of the underlying mortgage loans. If each borrower continues to make constant monthly payments until the maturity of their loan, the investor will also receive fixed monthly cash flows. However, in typical mortgage contracts, borrowers have the option to repay the loan principal and interest ahead of schedule without incurring additional costs. This behavior is called prepayment. Due to such prepayment behavior, the cash flows from MBSs are not constant over time and become more complex. Borrower's prepayment behavior is uncertain and depends on various macroeconomic factors, among which interest rates play a particularly significant role. When the prevailing mortgage rate $m(t)$ at time t falls below the fixed mortgage rate m on the borrower's existing loan, the borrower has an incentive to refinance. Accordingly, a mortgage pool with a positive difference $m - m(t)$ is more likely to experience prepayments. This difference $m - m(t)$ is referred to as moneyness. An MBS with prepayment risk can be interpreted as a position that is long an MBS with no prepayment and short an interest rate option. Since the increase of MBS prices in response to falling interest rates is limited compared to that of an MBS with no prepayment risk, MBSs exhibit negative convexity. While a large portion of observed prepayment rates can be explained by the refinancing incentive described above, other factors also contribute to prepayment behavior. As there are no liquid derivative instruments available to hedge prepayment risk, MBS investors generally bear this exposure without hedging. Modeling prepayment risk is important for MBS pricing, computing risk metrics, and constructing trading strategies. However, no unique "correct" prepayment model exists, and the model should be selected according to its intended application and the time horizon under consideration. This paper adopts a prepayment model that captures both interest rate-driven refinancing incentives and other sources of prepayment behavior.

Agency MBSs are traded either in the TBA (to-be-announced) market or the SP (specified-pool) market. The TBA market, which is a forward market, is particularly liquid. Figure 2 shows the trend in average daily trading volume by bond type in the United States [1] . From this figure, it can be seen that TBA trading of Agency MBSs represents a highly liquid market, with a trading volume approximately half that of U.S. Treasuries and significantly higher than that of SP trading of Agency MBSs. The following section explains the trading schedule and cash flow structure of 30-year Fannie Mae MBSs in the TBA market, which is the focus of this paper. Table 1 presents a hypothetical outline of a 30-year Fannie Mae MBSs. TBA trades are settled according to a schedule established by the Securities Industry and Financial Markets Association (SIFMA). On the trade date of a TBA

[1] The data is obtained from SIFMA (https://www.sifma.org/resources/archive/research/statistics/).

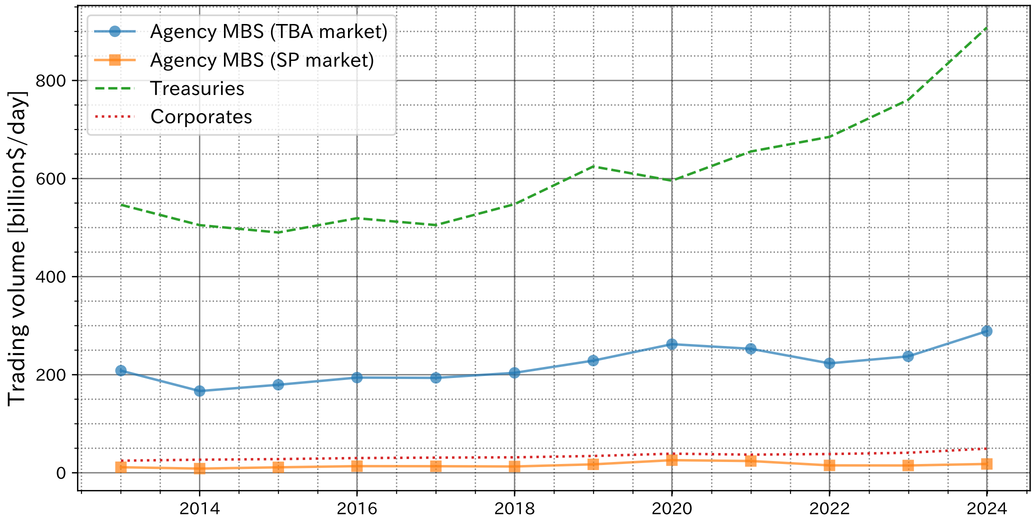

Figure 2: Trends in average daily trading volume by bond type in the United States. The circular markers (blue) and square markers (orange) represent trading volumes in the TBA and SP markets for Agency MBSs, respectively. The dashed line (green) and dotted line (red) represent trading volumes for U.S. Treasury securities and U.S. corporate bonds, respectively. The unit on the vertical axis is billion$/day. The data is obtained from SIFMA.

Table 1: Outline of cash flows for a hypothetical 30-year Fannie Mae MBSs traded in the TBA market. t_s denotes the settlement date; $t_i\,(i = 1, 2, \cdots)$ denotes the principal and interest payment dates; and $\tau_i\,(i = 1, 2, \cdots)$ indicates the notification dates on which the pool factors (prepayment rates) necessary to calculate the cash flows at t_i are announced. The table is adapted from Chernov, B. R. Dunn, and Longstaff (2017).

Date	Event	Time	Notes
3/31	Trade date	0	The buyer and seller agree on issuer, maturity, coupon, face value, and price. The pool is not specified at this point.
4/12	Notification date		The seller notifies the buyer of the specific MBS pool to be delivered at settlement.
4/14	Settlement date	t_s	The buyer pays the seller the agreed price plus accrued interest based on the notified pool.
4/30	Record date	τ_1	The buyer's holding is recorded in Fedwire.
5/7	Factor announcement date		Fannie Mae announces the pool factor reflecting April's prepayments.
5/25	Payment date	t_1	The buyer receives the first principal and interest payment.
5/31	Month-end	τ_2	The amount payable at t_2 reflects prepayments between τ_1 and τ_2.
6/26	Payment date	t_2	The buyer receives the next principal and interest payment.
⋮	⋮	⋮	⋮

transaction, the buyer and seller agree on six terms: price, par amount (face value), settlement date, issuer (agency program), maturity (mortgage type), and mortgage rate (Vickery and Wright 2013). Most TBA trades are forward contracts with settlement dates within three months, typically selected as one, two, or three months from the trade date. For 30-year Fannie Mae MBSs, the settlement date is usually scheduled for the second week of the month. On the notification date—set two business days prior to the settlement date—the seller informs the buyer of the specific pool to be delivered. This convention is known as the 48-hour rule. In general, the seller delivers the cheapest-to-deliver security from among the eligible pools, a practice similar to that in Treasury futures markets. On the settlement date, payment is made and the designated pool is delivered. As shown in Table 1, for each principal and interest payment date t_i $(i = 1, 2, \cdots)$, the corresponding pool factor (prepayment rate) necessary for calculating the payment is announced on date τ_i $(i = 1, 2, \cdots)$, which is set approximately one month in advance of t_i. In contrast to TBA trades, specified pool (SP) trades involve MBSs that are ineligible for TBA delivery or have special premium characteristics. In this paper, we estimate a model using forward prices observed in the TBA market and construct a corresponding trading strategy.

1.2 Related Literature

Early pricing models for MBSs are static and do not incorporate stochastic processes. However, K. B. Dunn and McConnell (1981a), K. B. Dunn and McConnell (1981b), and Brennan and Schwartz (1985) propose the Contingent Claims Model, which describes uncertainties related to interest rates and prepayment using stochastic processes and analyzes MBS pricing under a no-arbitrage framework. These models are further extended by K. B. Dunn and Spatt (2005) and Stanton and Wallace (1998). A key limitation of the above contingent claims models is that the cash flows and MBS prices generated by the models can substantially deviate from those observed in the market.

Schwartz and Torous (1989), Schwartz and Torous (1992), Schwartz and Torous (1993), Richard and Roll (1989), and Deng, Quigley, and Van Order (2000) propose econometric models based on historical data that incorporate regional and seasonal effects, burnout effects, and other macroeconomic factors. These models generate interest rate paths under the risk-neutral measure to compute cash flows. One limitation of this approach is that, while it can describe interest-rate-driven prepayment behavior, it fails to capture other factors. Another issue is that the characteristics such as prepayment rates differ across models, and the MBS prices produced by these models can substantially deviate from market prices. Carlin, Longstaff, and Matoba (2014) state that the predicted prepayment rates can vary significantly across dealers. Chernov, B. R. Dunn, and Longstaff (2017) also point out that even for a single dealer, the OAS can differ markedly before and after a model change, with differences exceeding 50 bps in some cases.

Other models include the arbitrage-free pricing frameworks known as the implied-prepayment model or the break-even-prepayment model, as discussed in Cheyette (1996) and Chaudhary (2006). These models estimate parameters using market prices, allowing the prepayment rates implied by the market to differ from the actual observed prepayment rates—an advantage of the approach. A limitation of the models in the above literature is that they cannot simultaneously explain the cross section of MBS prices with different mortgage rates. Levin and Davidson (2005) discuss a method for explaining the cross section of MBS prices across different mortgage rates. Chernov, B. R. Dunn, and Longstaff (2017) demonstrate that an arbitrage-free pricing model can explain the cross section of multiple MBS prices with high accuracy, and B. Dunn, Kargar, and Zheng (2021) extend their model to study liquidity risk in the MBSs market.

When constructing models for MBSs, the choice of pricing and prepayment models should depend on their intended application. For dealers at securities firms who require pricing of individual pools, it is appropriate to use complex prepayment models estimated with a wide range of explanatory variables—such as the econometric models mentioned above. In contrast, institutional investors who use the TBA market to manage individual MBS positions may prefer prepayment models that are relatively simple but fit TBA market prices well. This paper adopts the perspective of institutional investors who trade MBSs in the TBA market and manage positions over at least a monthly time horizon. Accordingly, we adopt the arbitrage-free pricing model proposed by Chernov, B. R. Dunn, and Longstaff (2017), which provides a good fit to both the cross section and time series of TBA market prices.

Regarding prior research on MBS trading strategies, Duarte, Longstaff, and Yu (2006) construct five types of fixed income arbitrage strategies: swap spread arbitrage, yield curve arbitrage, mortgage arbitrage, volatility arbitrage, and capital structure arbitrage. They report that the mortgage arbitrage strategy delivers the highest return among them. Diep, Eisfeldt, and Richardson (2021) propose an asset pricing model for the cross section of MBS returns. Gabaix, Krishnamurthy, and Vigneron (2007), Boyarchenko, Fuster, and Lucca (2019), and Diep, Eisfeldt, and Richardson (2021) examine the prepayment risk premium that is believed to exist in expected MBS returns. To the best of our knowledge, no existing study has employed an arbitrage-free pricing model to construct an MBS trading strategy. This paper addresses whether there remain persistent mispricings by constructing a fixed income arbitrage strategy based on an arbitrage-free pricing model.

The remainder of the paper is organized as follows. Section 2 describes the MBS valuation model —covering the MBS cash flows, the price of the MBS itself, the prepayment process, and the MBS forward price. Section 3, section 4 present the estimation methodology and the estimation results respectively. In Section 5, we construct an arbitrage trading strategy based on the estimated model and evaluate its performance. Section 6 concludes.

2 Valuation Model

In this section, we explain the valuation model of a pass-through type MBS used in this paper. First, subsection 2.1 explains the cash flow of an MBS considering the prepayment of mortgage loans. Next, in subsection 2.2, in addition to the interest rate model, we discuss the theoretical cash price of a pass-through MBS, taking into account the default risk of the MBS issuer. In subsection 2.3, we discuss the characterization of prepayment risk and present our prepayment process. Finally, we derive a theoretical formula for the forward price quoted in the TBA market in subsection 2.4 so that we can adapt the TBA forward price formula to actual market practices.

We will introduce some stochastic models throughout this section, so we firstly introduce a probability space $(\Omega, \mathcal{F}, \mathbb{Q})$, where $\mathbb{Q}$ denotes the risk-neutral probability measure. In short, we assume that the market is arbitrage-free and one pricing measure is arbitrarily fixed. All the stochastic components are supposed to be defined on this probability space. We also let $\{\mathcal{F}_t\}$ denote a complete filtration generated by several Brownian motions, whose specific form will be described later. This filtration will be interpreted as the market information filtration that does not include default information.

2.1 Cash Flow of MBS

This subsection outlines the time evolution of cash flows from mortgage loans, which serve as the underlying assets for MBSs, and then derives the expression for MBS cash flows in the presence of prepayment.

We consider a pass-through type MBS with maturity T and the constant repayment rate c, defined as the total payment rate per unit time from the underlying collateral loan pool (hereafter referred to as the principal and interest payment rate). For the sake of discussing a theoretical model for a while, we assume that the MBS holder continuously receives the cash flows from the collateral loan pool. In subsection 2.4, we will present a discrete-time formulation of cash flows to be consistent with market conventions of the actual TBA market.

First, we consider the case without any prepayment from the underlying collateral loan pool. We assume that the mortgage rate contracted by the borrowers in the underlying loan pool is a fixed rate m over the period $[0, T]$. Denote by $\tilde{P}(t)$ the remaining principal balance scheduled at time t without prepayment. In this case, the MBS cash flow over the infinitesimal time interval $(t, t + \mathrm{d}t]$ is given by $c\,\mathrm{d}t$, which equals the sum of the principal reduction and the interest payment during the same period. Therefore, the following relationship holds.

$$c\,\mathrm{d}t = -\mathrm{d}\tilde{P}(t) + \tilde{P}(t)\cdot m\,\mathrm{d}t. \tag{1}$$

By solving this differential equation under the terminal condition $\tilde{P}(T)=0$, that is, the principal is fully repaid at maturity T, and $\tilde{P}(0)=1$, that is, the normalization condition that the initial principal is unity, it follows

$$\tilde{P}(t) = \frac{1-e^{-m(T-t)}}{1-e^{-mT}}. \tag{2}$$

At the same time, we obtain the following relation.

$$1 = \int_0^T e^{-mt}\cdot c\,\mathrm{d}t \quad \Leftrightarrow c = \frac{m}{1-e^{-mT}}. \tag{3}$$

This can be regarded as the counterpart of the relationship between maturity and coupon rate for a par bond in the context of Treasury securities.

Next, we move the case where the collateral loan pool involves prepayment. Assume that, over an infinitesimal time interval $(t, t+\mathrm{d}t]$, an amount $\lambda(t)\cdot \mathrm{d}t$ per unit of initial principal can be prepaid. Hereafter, we refer to $\lambda(t)$ as the prepayment process, which will be specified by a nonnegative $\{\mathcal{F}_t\}$-adapted process. We discuss how the prepayment process is specifically modeled later in subsection 2.3, and for the current discussion, we may treat the prepayment process $\lambda(t)$ like a deterministic function of time t.

Let $P(t)$ denote the remaining principal balance at time t in the presence of prepayment. The change in the remaining principal balance over the infinitesimal time interval $(t, t+\mathrm{d}t]$ is considered as the sum of the following two components: (i) the scheduled change in the remaining principal in the absence of prepayment, applied to the already reduced balance due to prior prepayments, and (ii) the new prepayment happened over the interval.

Therefore, we have the following equation,

$$\mathrm{d}P(t) = \frac{P(t)}{\tilde{P}(t)}\,\mathrm{d}\tilde{P}(t) - \lambda(t)P(t)\,\mathrm{d}t.$$

Solving this equation for $P(t)$ with the initial principal normalized to unity yields

$$P(t) = e^{-\int_0^t \lambda(u)\,\mathrm{d}u}\tilde{P}(t). \tag{4}$$

Remark that the remaining principal balance $P(t)$ actually becomes a stochastic process because the prepayment process $\lambda(t)$ is specified in terms of some stochastic components later.

From the above, we can consider the cash flows that the MBS holder can receive under the assumption that the MBS issuer does not default. Let $\mathrm{CF}(t)$ be the cumulative cash flow received by the MBS holder up to time t. The infinitesimal cash flow $\mathrm{dCF}(t)$ over the interval $(t, t+\mathrm{d}t]$ can be regarded as the sum of the scheduled principal and interest payment, given by $e^{-\int_0^t \lambda(u)\,\mathrm{d}u}c\,\mathrm{d}t$, and the principal repayment due to prepayment during that interval, $P(t)\lambda(t)\,\mathrm{d}t$.

By using Equations (2) and (4), the infinitesimal cash flow $\mathrm{dCF}(t)$ can be expressed as:

$$\mathrm{dCF}(t) = e^{-\int_0^t \lambda(u)\,\mathrm{d}u}\left(c + \frac{1-e^{-m(T-t)}}{1-e^{-mT}}\lambda(t)\right)\mathrm{d}t. \tag{5}$$

Figure 3 illustrates the time evolution of the three components of MBS cash flows, namely scheduled principal payments, scheduled interest payments, and principal payments resulting from prepayment, under several different prepayment processes $\lambda(t)$. The first row of Figure 3 shows the cash flows when $\lambda(t)\equiv 0$, which means there is no prepayment. In this case, there are no principal payments due to prepayment. While the total of scheduled principal and interest payments remains constant over time,

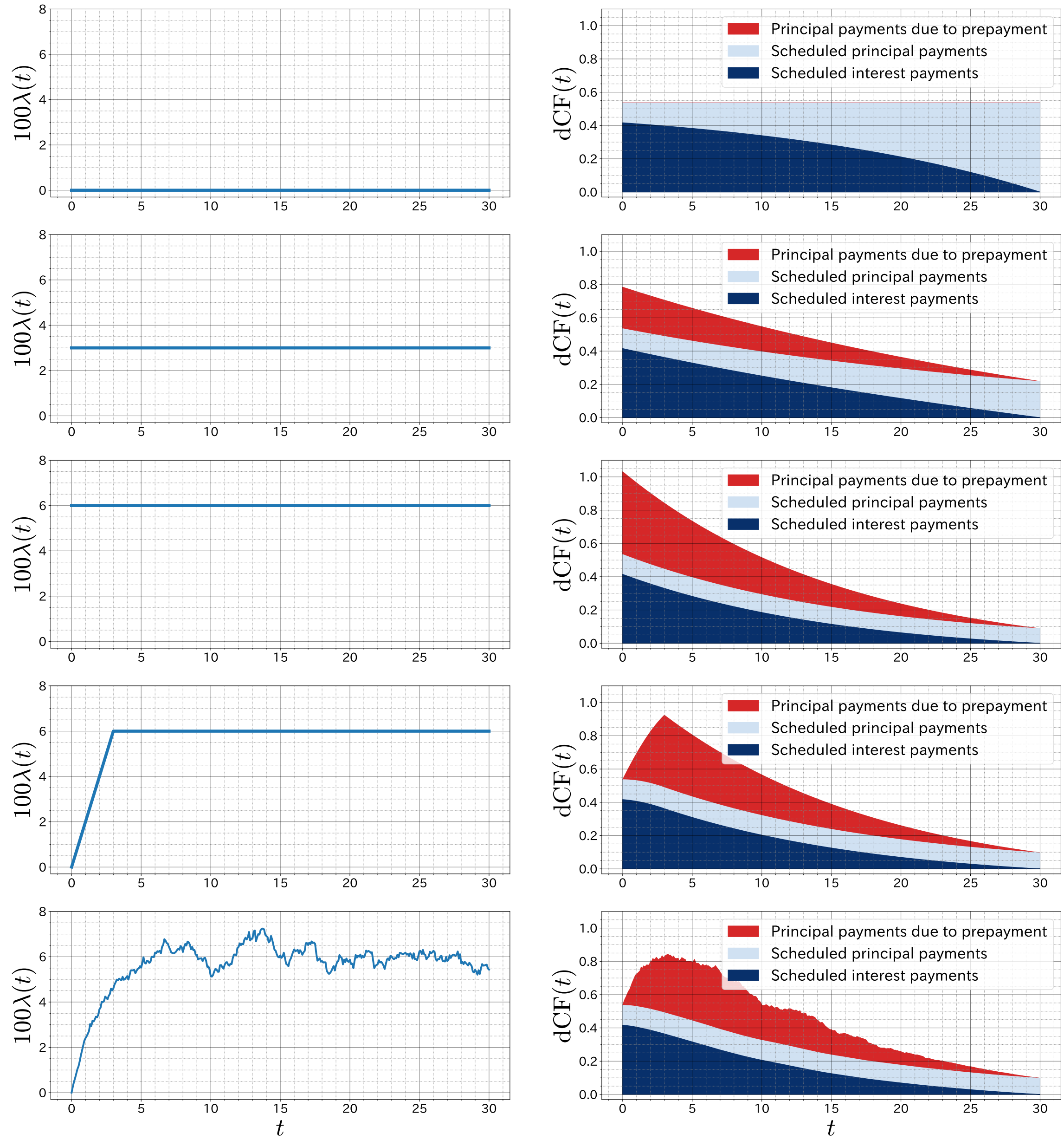

Figure 3: Time evolution of the three components of MBS cash flows, namely scheduled principal payments, scheduled interest payments, and principal payments resulting from prepayment, under several different prepayment processes $\lambda(t)$. (Left panel) Prepayment process $\lambda(t)$. (Right panel) Corresponding cash flow $\mathrm{dCF}(t)$ associated with the prepayment process shown in the left panel. (Row 1) Cash flows in the case of no prepayment, i.e., $\lambda(t) \equiv 0$. (Row 2) Cash flows when $\lambda(t)$ is constant at $3.0\% = 0.03$. (Row 3) Cash flows when $\lambda(t)$ is constant at $6.0\% = 0.06$. (Row 4) Cash flows when $\lambda(t)$ follows the PSA model. (Row 5) Cash flows when $\lambda(t)$ is stochastic.

the composition changes. In the early periods, principal payments dominate the cash flows, whereas near maturity, the amount of principal payments decreases and the share of interest payments becomes larger. In this way, MBSs differ from typical government or corporate bonds in that principal is paid down gradually throughout the life of the security. The second through fifth rows of Figure 3 illustrate the time evolution of cash flows under different prepayment processes. The second and third rows correspond to constant prepayment rates of $\lambda(t) \equiv 3.0\% = 0.03$ and $\lambda(t) \equiv 6.0\% = 0.06$, respectively. When the prepayment rate is higher, more principal is paid earlier, which results in a faster reduction in the remaining balance and a quicker decline in interest payments. The fourth row of Figure 3 presents the cash flows under the PSA model, which is a prepayment function frequently used in the industry (For the PSA model, see for example Frank J. Fabozzi 2008). The fifth row shows the cash flows generated by a sample path of a stochastic prepayment process $\lambda(t)$, which will be discussed later. In each case, the time profile of principal payments due to prepayment displays distinctive patterns. Across all panels in Figure 3, we observe that whereas the principal of government and corporate bonds is paid at maturity, MBSs amortize principal gradually over the life of the instrument, resulting in a more even distribution of cash flows. This indicates that MBSs are exposed to interest rate risk across the entire yield curve.

2.2 MBS Price

MBSs can be seen as the financial instrument that embeds both interest rate risk and the associated prepayment risk. In addition, credit risk related to the possibility of default by the issuer, Fannie Mae, should also be taken into account. Therefore, in order to evaluate the theoretical price of MBSs, it is necessary to prepare some continuous-time stochastic models that capture these risks. In this subsection, following the discussion of cash flow formulation in Section 2.1, we describe the theoretical expression for the MBS price that accounts for latent variables other than cash flow variation arising from prepayment risk. The modeling of prepayment risk will be discussed in the next Section 2.3.

We first adopt the following Hull-White model (Hull and White 1990) as the short-rate interest rate model,

$$\mathrm{d}r(t) = [\alpha_r(t) - \beta_r r(t)]\ \mathrm{d}t + \sigma_r\, \mathrm{d}W_r^{\mathbb{Q}}(t). \tag{6}$$

Here, $W_r^{\mathbb{Q}}$ denotes a standard Brownian motion under $\mathbb{Q}$ with respect to the filtration $\{\mathcal{F}_t\}$. The function $\alpha_r(t)$ is defined using the discount curve observed in the market, $D^{\mathrm{M}}(t)$, at the current time $t = 0$ in the following way,

$$\alpha_r(t) = -\frac{\partial^2 \ln D^{\mathrm{M}}(t)}{\partial t^2} - \beta_r \frac{\partial \ln D^{\mathrm{M}}(t)}{\partial t} + \frac{\sigma_r^2}{2\beta_r}\left(1 - e^{-\beta_r t}\right). \tag{7}$$

When $\alpha_r(t)$ satisfies Equation (7), the theoretical discount curve based on the short-rate process defined by Equation (6) coincides with the discount curve observed in the market (see Kijima and Nagayama 1994 for a proof).

Next, we consider the credit risk embedded in MBSs. Whether the cash flows from Fannie Mae MBSs are guaranteed by the U.S. government remains a subject of ongoing discussion, and this study focuses specifically on these securities. For Ginnie Mae MBSs, the cash flows are fully guaranteed by the U.S. government, and therefore carry the same level of safety as U.S. Treasury securities. In contrast, the cash flows from Fannie Mae MBSs are guaranteed not by the government but by the GSE itself. For investors who believe that the government will intervene in times of crisis, this GSE guarantee may be perceived as an implicit government guarantee. Indeed, on September 6, 2008, the U.S. government announced that Fannie Mae and Freddie Mac[2] would be placed into conservatorship under the Federal Housing Finance Agency (FHFA), a government agency. Since then, it has been argued that the conservatorship implicitly provides a government guarantee on the obligations of these GSEs. On the other hand, one could argue that this government support was temporary and does not amount to an explicit guarantee comparable to that of Ginnie Mae. In fact, current prospectuses

[2]The Federal Home Loan Mortgage Corporation.

for MBSs issued by Fannie Mae clearly state that the cash flows are not guaranteed by the U.S. government.

Thus, opinions vary as to whether the cash flows of Fannie Mae MBSs are guaranteed to the same extent as those of U.S. Treasuries. Taking this into account, this paper adopts a modeling framework in which a positive credit spread over Treasuries is allowed for MBSs. In other words, we construct an MBS model in which there is a risk that cash flows will not be received due to the possible default of Fannie Mae.

Let τ be a non-negative random variable representing the default time of Fannie Mae, the issuer of the MBS. We define the default information filtration $\{\mathcal{H}_t\}$ and the investor information filtration $\{\mathcal{G}_t\}$ (both assumed to be completed) as follows,

$$\mathcal{H}_t = \sigma\left\{\tau \wedge s \mid s \leq t\right\},$$
$$\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{H}_t.$$

With this definition, τ is a $\{\mathcal{G}_t\}$-stopping time. The hazard rate process associated with the $\{\mathcal{G}_t\}$-stopping time τ is represented by an $\{\mathcal{F}_t\}$-predictable process $\{w(t)\}_{0\leq t\leq T}$. For $s > t$, we assume the following relationship holds,

$$\mathbb{Q}\left(\tau > s \mid \mathcal{G}_t\right) = \mathbb{1}_{\{\tau>t\}}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^s w(u)\,\mathrm{d}u}\middle|\,\mathcal{F}_t\right].$$

Default risk involves not only the possibility of default occurrence but also the magnitude of loss in the event of default. For example, Duffie and Singleton (Duffie and Singleton 1999) show under various assumptions that the relationship "credit spread = hazard rate × loss given default" holds. Here, we assume the loss given default is 100%, meaning the recovery rate at default is zero. Under this assumption, the credit spread can be regarded as the hazard rate itself.

As the model for the hazard rate process, we adopt the following CIR process (Cox, Ingersoll, and Ross 1985),

$$\mathrm{d}w(t) = \left[\alpha_w - \beta_w w(t)\right]\,\mathrm{d}t + \sigma_w\sqrt{w(t)}\,\mathrm{d}W_w^{\mathbb{Q}}(t). \tag{8}$$

We assume the hazard rate process is independent of the interest rate process and other latent processes described later. It is known that this assumption has little effect on the results (see Duffie and Singleton 1997, Longstaff, Mithal, and Neis 2005, Pan and Singleton 2008, Longstaff, Pan, et al. 2011, and Ang and Longstaff 2013). Under the CIR-type hazard rate process, the survival probability function $S(t) := \mathbb{Q}(\tau > t)$ is calculated as follows (for example, see Brigo and Mercurio 2013),

$$S(t) = \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t w(s)\,\mathrm{d}s}\right] = e^{A_w(t)-B_w(t)w(0)}, \tag{9}$$

$$A_w(t) = \frac{2\alpha_w}{\sigma_w^2}\ln\left[\frac{2h\,e^{(\beta_w+h)t/2}}{2h+(\beta_w+h)(e^{ht}-1)}\right], \tag{10}$$

$$B_w(t) = \frac{2(e^{ht}-1)}{2h+(\beta_w+h)(e^{ht}-1)}. \tag{11}$$

For the derivation of the second equality in Equation (9), refer to Section A.1.

Based on the above, the present value of the MBS, denoted by $V^{\mathrm{MBS}}(0)$, under the assumption that the recovery rate upon default of the issuer is zero, is given by the following expression,

$$V^{\mathrm{MBS}}(0) = \mathbb{E}^{\mathbb{Q}}\left[\int_0^T e^{-\int_0^t [r(u)+w(u)]\,\mathrm{d}u}\,\mathrm{dCF}(t)\right]. \tag{12}$$

For additional explanation of Equation (12), refer to Section A.3. The cumulative cash flow $\mathrm{CF}(t)$, conditional on no default of the MBS issuer up to time t, is assumed to be an $\{\mathcal{F}_t\}$-predictable stochastic process. The modeling of prepayment risk, which affects the cumulative cash flow, is discussed in Section 2.3. Furthermore, due to the assumed independence between the hazard rate process and other latent variables such as the interest rate process and the cash flow process, the MBS price can be rewritten using the survival probability function $S(t)$ as follows,

$$V^{\mathrm{MBS}}(0) = \int_0^T S(t)\, \mathbb{E}^{\mathbb{Q}} \left[e^{-\int_0^t r(u)\, \mathrm{d}u}\, \mathrm{dCF}(t) \right]. \tag{13}$$

2.3 Prepayment Process

In this section, we discuss the characterization of prepayment risk, which is the most important source of risk in MBS valuation as it affects the variability of cash flows. Specifically, we examine how the prepayment process $\lambda(t)$, introduced in Equation (4), should be formulated.

We define the mortgage rate of TBA forward contract i as m_i, and the Fannie Mae US 30-Year Fixed Rate Mortgage at time t as $m(t)$. Then moneyness is defined as $m_i - m(t)$. Figure 4 shows a scatter plot of moneyness and 1-month CPR for Fannie Mae 30-year MBSs with mortgage rates of 2.5%, 3.0%, $\cdots$, and 5.0%. From Figure 4, we observe that the 1-month CPR tends to be close to zero

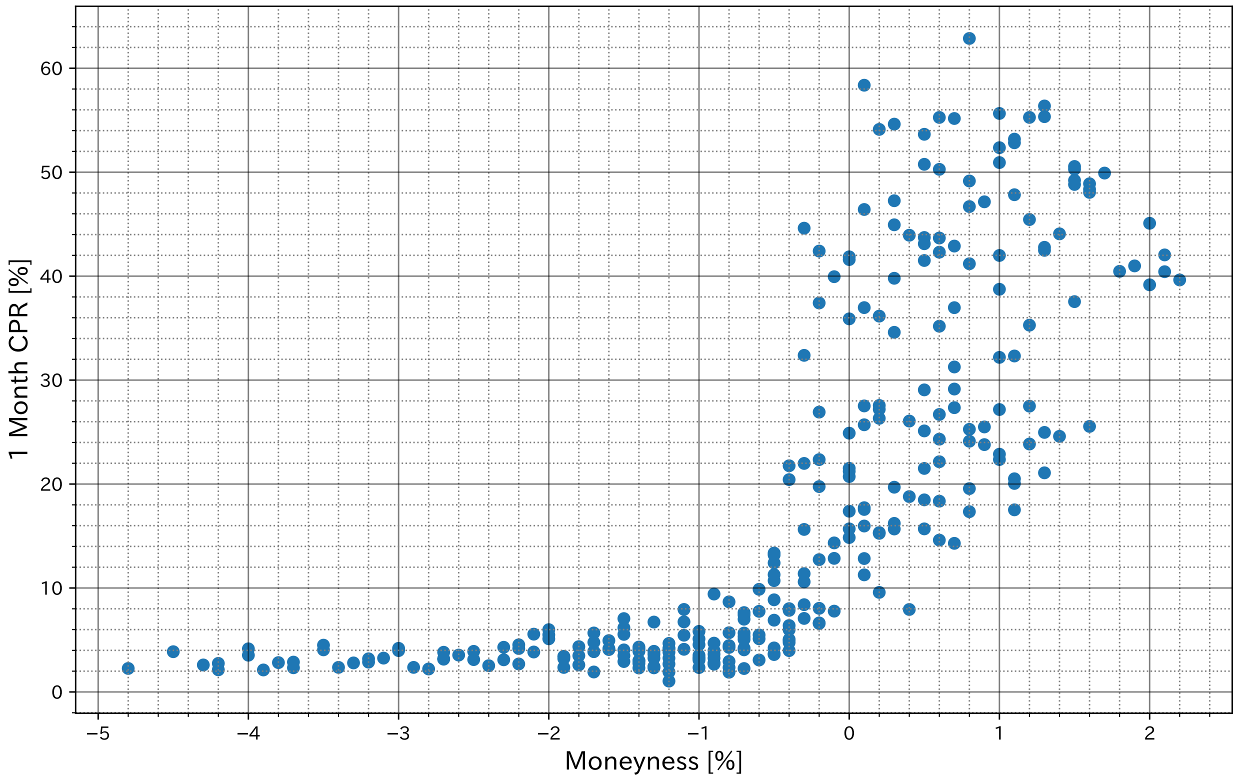

Figure 4: Scatter plot of moneyness and 1-month CPR for Fannie Mae 30-year MBSs with mortgage rates of 2.5%, 3.0%, $\cdots$, and 5.0%. The sample period is from June 2012 to March 2024, and the data frequency is quarterly (end of March, June, September, and December). The Fannie Mae US 30-Year Fixed Rate Mortgage used to compute moneyness and the 1-month CPR are obtained from Bloomberg.

when moneyness is in the negative region, and increases approximately linearly when moneyness is in the positive region. Figure 5 shows a scatter plot of the US 10-year Treasury zero yield and the Fannie Mae US 30-Year Fixed Rate Mortgage. From Figure 5, we observe that the US 10-year Treasury zero yield and the Fannie Mae US 30-Year Fixed Rate Mortgage exhibit a very strong positive correlation,

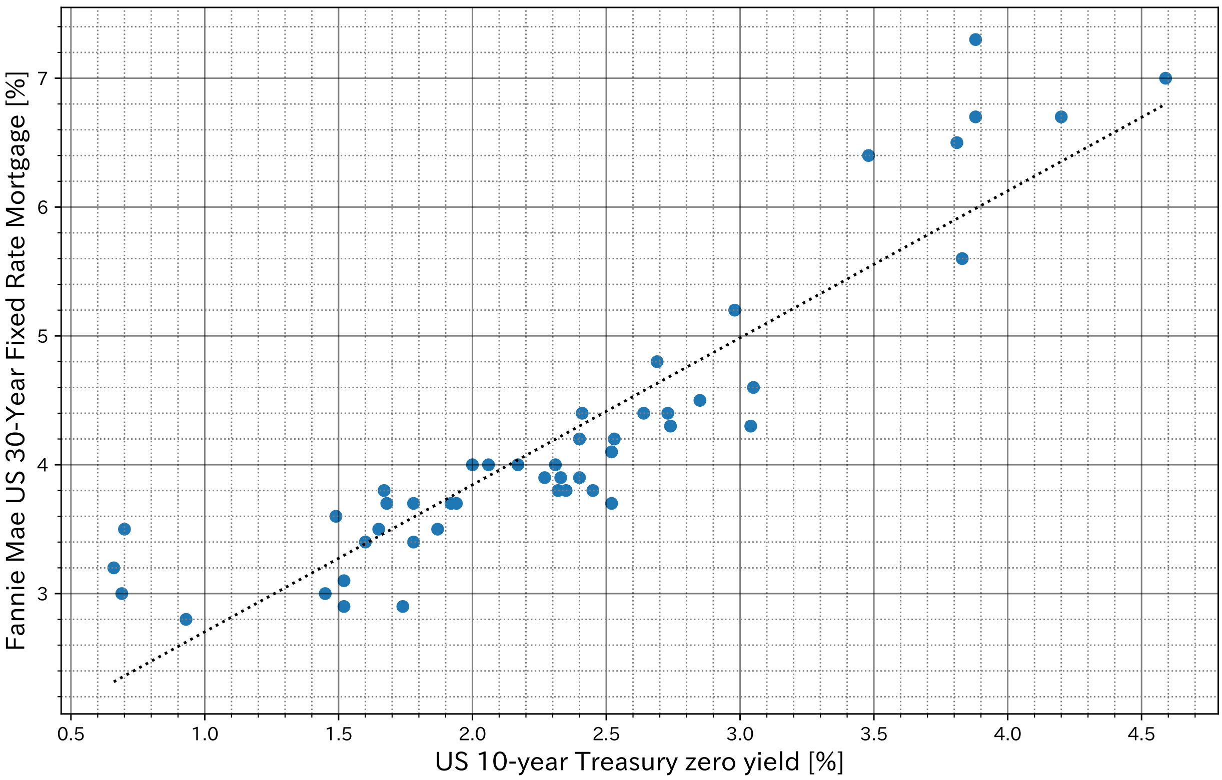

Figure 5: Scatter plot of the US 10-year Treasury zero yield and the Fannie Mae US 30-Year Fixed Rate Mortgage. The correlation coefficient between the US 10-year Treasury zero yield and the Fannie Mae 30-year mortgage rate is 0.91. The black dashed line represents the estimated simple linear regression line, where the dependent variable is the Fannie Mae 30-year mortgage rate and the explanatory variable is the US 10-year Treasury zero yield. The estimated slope is 1.14, and the intercept is 1.56%. The sample period is from June 2012 to March 2024, and the data frequency is quarterly (end of March, June, September, and December). The Fannie Mae 30-Year Fixed Rate Mortgage is obtained from Bloomberg, and the US Treasury zero yield data is obtained from the Federal Reserve website (https://www.federalreserve.gov/releases/h15/).

with a correlation coefficient of 0.91. In addition, the intercept of the simple linear regression line—where the dependent variable is the Fannie Mae 30-year mortgage rate and the explanatory variable is the US 10-year Treasury zero yield—is estimated to be 1.56%. Based on this observation, we use $a + bR_{10}(t)$ as a proxy for the Fannie Mae 30-year fixed mortgage rate $m(t)$ when modeling the prepayment rate. Here, $R_{10}(t)$ denotes the 10-year Treasury zero yield, and a and b are constants. Under this specification, moneyness is represented as $m_i - a - bR_{10}(t)$. Note that the 10-year zero yield and the short-rate process satisfy the following relationship,

$$
\begin{aligned}
R_{10}(t) &= \frac{1}{10}\left[-A_r(t) + B_r r(t)\right], \\
\text{where} \quad A_r(t) &= \ln\frac{D^{\mathrm{M}}(t+10)}{D^{\mathrm{M}}(t)} - B_r\frac{\partial \ln D^{\mathrm{M}}(t)}{\partial t} - \frac{\sigma_r^2}{4\beta_r}\left(1 - e^{-2\beta_r t}\right)B_r^2, \\
B_r &= \frac{1 - e^{-10\beta_r}}{\beta_r}.
\end{aligned}
$$

Note that $R_{10}(t)$ does not represent the 10-year zero yield at the current time $t = 0$, but rather the 10-year zero yield at time $t \geq 0$.

Based on the above considerations, we specify the prepayment process $\lambda_i(t)$ with mortgage rate m_i as follows in this paper,

$$
\lambda_i(t) = x(t) + y(t)\cdot\max\left\{0, m_i - a - bR_{10}(t)\right\}.
$$

The processes $x(t)$ and $y(t)$ can be interpreted as latent factors related to prepayment, where $x(t)$ represents the component independent of moneyness, and $y(t)$ captures the sensitivity to moneyness. Alternatively, $x(t)$ may be interpreted as a latent factor common to all MBSs, and $y(t)$ as a latent factor dependent on the mortgage rate of each MBS. By defining the prepayment process in terms of the 10-year zero yield $R_{10}(t)$ rather than the Fannie Mae US 30-Year Fixed Rate Mortgage, we are able to compute values of $\max\{0, m_i - a - bR_{10}(t)\}$ for each interest rate path generated from a properly estimated interest rate model. Since the observed prepayment rate is non-negative by definition, we assume that the latent processes $x(t)$ and $y(t)$ follow the CIR model (Cox, Ingersoll, and Ross 1985),

$$
\mathrm{d}x(t) = \left[\alpha_x - \beta_x x(t)\right]\mathrm{d}t + \sigma_x\sqrt{x(t)}\mathrm{d}W_x^{\mathbb{Q}}(t), \tag{14}
$$

$$
\mathrm{d}y(t) = \left[\alpha_y - \beta_y y(t)\right]\mathrm{d}t + \sigma_y\sqrt{y(t)}\mathrm{d}W_y^{\mathbb{Q}}(t). \tag{15}
$$

Here, $W_x^{\mathbb{Q}}$ and $W_y^{\mathbb{Q}}$ are assumed to be standard Brownian motions under the risk-neutral probability measure $\mathbb{Q}$ with respect to the filtration $\{\mathcal{F}_t\}$. Since it is reasonable to consider that the prepayment process is linked to the short rate, we denote the correlation between $W_r^{\mathbb{Q}}$ and $W_x^{\mathbb{Q}}$ as $\rho_{r,x}$, and the correlation between $W_r^{\mathbb{Q}}$ and $W_y^{\mathbb{Q}}$ as $\rho_{r,y}$. We also allow for the possibility that $x(t)$ and $y(t)$ are correlated, and denote the correlation between $W_x^{\mathbb{Q}}$ and $W_y^{\mathbb{Q}}$ as $\rho_{x,y}$. Given these assumptions, the market information filtration can be regarded as the natural filtration generated by the four standard Brownian motions $W_r^{\mathbb{Q}}$, $W_w^{\mathbb{Q}}$, $W_x^{\mathbb{Q}}$, and $W_y^{\mathbb{Q}}$.

Therefore, combining Equations (5) and (13) with the prepayment process model described above, the current price of an MBS with mortgage rate m_i is given by the following expression,

$$
\begin{aligned}
V_i^{\mathrm{MBS}}(0) &= \int_0^T S(t)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^t (r(u)+\lambda_i(u))\mathrm{d}u}\left(c + \frac{1 - e^{-m_i(T-t)}}{1 - e^{-m_i T}}\lambda_i(t)\right)\right]\mathrm{d}t \\
&= \int_0^T S(t)\mathbb{E}^{\mathbb{Q}}\Bigg[\exp\left(-\int_0^t \left[r(u) + x(u) + y(u)\cdot\max\left\{0, m_i - a - bR_{10}(u)\right\}\right]\mathrm{d}u\right) \\
&\qquad \times\left(c + \frac{1 - e^{-m_i(T-t)}}{1 - e^{-m_i T}}\left[x(t) + y(t)\cdot\max\left\{0, m_i - a - bR_{10}(t)\right\}\right]\right)\Bigg]\mathrm{d}t.
\end{aligned} \tag{16}
$$

2.4 MBS Forward Price

In the preceding subsection, we have discussed the models of the state variables used for MBS pricing and the corresponding theoretical prices. However, since what is actually quoted in the TBA market, which is the subject of our analysis, is not the price of an MBS itself but the forward price, the estimation of the model using TBA market prices requires the theoretical forward price of an MBS. This theoretical forward price is derived in B. Dunn, Kargar, and Zheng (2021), and we present the expression below.

When evaluating the TBA market, we consider the case in which MBS cash flows are discrete in time, consistent with actual market conventions. In that case, the MBS price is given by the following expression, which is a discretized version of Equation (13) with respect to time,

$$V^{\mathrm{MBS}}(0) = \sum_{k=1}^{K} S(t_k)\, \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_k} r(s)\,\mathrm{d}s}\mathrm{CF}(t_k)\right].$$

The discrete cash flows $\mathrm{CF}(t_k)\,(k = 1, \cdots, K)$ are obtained by discretizing the continuous-time expression given in Equation (5) and replacing continuously compounded interest with monthly compounded interest. Let $\widetilde{\mathrm{PRIN}}(t_k)$, $\widetilde{\mathrm{INT}}(t_k)$, and $\mathrm{PP}(t_k)$ denote the scheduled principal payment, scheduled interest payment, and prepayment principal at time t_k $(k = 1, \cdots, K)$, respectively. Then, the cash flow $\mathrm{CF}(t_k)$ at time t_k is given by the following expression (see Section A.2 for the derivation),

$$\begin{aligned}
\mathrm{CF}(t_k) &= \widetilde{\mathrm{PRIN}}(t_k) + \frac{m-g}{m}\,\widetilde{\mathrm{INT}}(t_k) + \mathrm{PP}(t_k), && (17)\\
\text{where}\quad \mathrm{PP}(t_k) &:= \left[\widetilde{\mathrm{BAL}}(t_{k-1}) - \widetilde{\mathrm{PRIN}}(t_k)\right]\cdot \mathrm{SMM}(\tau_k), && (18)\\
\widetilde{\mathrm{PRIN}}(t_k) &:= N(\tau_{k-1})\cdot \mathrm{PRIN}(t_k), && (19)\\
\widetilde{\mathrm{INT}}(t_k) &:= N(\tau_{k-1})\cdot \mathrm{INT}(t_k), && (20)\\
\widetilde{\mathrm{BAL}}(t_k) &:= N(\tau_k)\cdot \mathrm{BAL}(t_k), && (21)\\
\mathrm{SMM}(\tau_k) &:= \frac{N(\tau_{k-1}) - N(\tau_k)}{N(\tau_{k-1})}, && (22)\\
\mathrm{INT}(t_k) &:= \mathrm{PAY}(t_k) - \mathrm{PRIN}(t_k), && (23)\\
\mathrm{PRIN}(t_k) &:= \mathrm{PAY}(t_k)\cdot\left(1+\frac{m}{12}\right)^{-K+k-1}, && (24)\\
\mathrm{BAL}(t_k) &:= \frac{1-\left(1+\frac{m}{12}\right)^{-K+k}}{1-\left(1+\frac{m}{12}\right)^{-K}}, && (25)\\
\mathrm{PAY}(t_k) &:= \frac{\frac{m}{12}}{1-\left(1+\frac{m}{12}\right)^{-K}} \quad (K := mT). && (26)
\end{aligned}$$

Here, $\mathrm{PAY}(t_k)$, $\mathrm{INT}(t_k)$, $\mathrm{PRIN}(t_k)$, and $\mathrm{BAL}(t_k)$ represent, respectively, the scheduled monthly payment, the interest portion of the payment, the principal portion of the payment, and the remaining balance at time t_k $(k = 1, \cdots, K)$ in the case without prepayment. The parameter g denotes the fee rate, which is the sum of the servicing fee and the guarantee fee. Also, τ_k $(k = 1, \cdots, K)$ denotes the time at which the cash flow amount at t_k is determined. For 30-year MBSs issued by Fannie Mae, τ_k typically occurs about one month before t_k, and this paper defines $\tau_k := t_k - \frac{1}{12}$ (see Figure 10 in the section on the derivation of Equations 17–26).

The theoretical forward price of an MBS depends on the number of underlying cash flow payments occurring between the current time 0 and the settlement date t_s. Accordingly, let $\mathrm{Fwd}(0, t_s)$ denote the forward price at the current time 0 for an MBS forward contract with settlement date t_s. The theoretical forward price of an MBS under discrete cash flows is then given by the following expression,

$$\mathrm{Fwd}(0,t_s) = \frac{100}{\mathrm{BAL}(t_s)\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_s} r(u)\mathrm{d}u}\right]}\left\{\mathbb{E}^{\mathbb{Q}}\left[e^{\int_0^{\tau_s}\lambda(u)\mathrm{d}u}\right]V^{(\mathrm{MBS})}(0)\right.$$
$$\left. - \sum_{i|t_i<t_s}\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_i}(r(u)+w(u))\mathrm{d}u+\int_0^{\tau_s}\lambda(u)\mathrm{d}u}\mathrm{CF}(t_i)\right]\right\}. \tag{27}$$

Here, we define the final cash flow determination date before the settlement date as

$$\tau_s := \mathbb{1}_{\{t_s \geq t_1\}} \max\left\{\tau_k \mid t_k < t_s \ (k = 1, 2, \cdots)\right\}.$$

Note that the prepayment history up to the time τ_s should be reflected in the calculation of the remaining principal balance. For additional details on the expression in Equation (27), see Section A.4.

The settlement date of the TBA forward contract for the 30-year Fannie Mae MBS, which is the focus of this paper, is typically scheduled in the second week of the settlement month. However, in this paper, we use the TBA forward price as of the month-end business day in our empirical analysis. Accordingly, the analysis employs three contract maturities, approximately 0.3, 1.3, and 2.3 months of remaining time to settlement, which are considered to have relatively high liquidity at the month-end business day (time 0). In particular, when $t_s = 0.3$ months, no cash flow occurs before the settlement date, and the TBA forward price is therefore given by the following expression,

$$\mathrm{Fwd}(t_0, t_\mathrm{s}) = \frac{100}{\mathrm{BAL}(t_s)}\frac{V^{(\mathrm{MBS})}(t_0)}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{t_s} r(s)\,\mathrm{d}s}\right]}. \tag{28}$$

3 Estimation Methodology

This section explains the estimation method for the latent state processes and the data used for the estimation. Section 3.1 describes the estimation method for the short rate process related to interest rates, and Section 3.2 presents the estimation methods for the credit spread process and the prepayment process using TBA forward contracts.

3.1 Short Rate Process

In this subsection, we describe the estimation method for the function $\alpha_r(t)$ and the parameters β_r and σ_r in the Hull-White model (7), which is used as the short rate model. The U.S. zero-coupon yield data are obtained from the Federal Reserve Board's website[3], using yields with maturities of 1 month, 3 months, 6 months, 1 year, 2 years, 3 years, 5 years, 7 years, 10 years, 20 years, and 30 years. The discount curve $D^{\mathrm{M}}(t)$, where t denotes the time to maturity, is constructed by applying cubic spline interpolation to the zero-yield data. The function $\alpha_r(t)$ is then obtained using Equation (7). The parameters β_r and σ_r are estimated using swaption data. Specifically, 15 at-the-money-forward European receiver swaptions are used, with option maturities $n \in \mathcal{T}_{\mathrm{mat}}^{\mathrm{opt}} := \{1, 2, 3, 4, 5\}$, and underlying swap tenors $m \in \mathcal{T}_{\mathrm{tnr}} := \{5, 7, 10\}$. The prices of those swaptions are denoted by $P_{n,m}^{\mathrm{M}}$, referred to as n-into-m swaption. The swaption data are obtained from Bloomberg, and the present value P^{M} is computed from the quoted normal volatilities σ_{N}. At each time, the parameters β_r and σ_r are determined by minimizing the following loss function,

$$\sum_{n\in\mathcal{T}_{\mathrm{mat}}^{\mathrm{opt}}}\sum_{m\in\mathcal{T}_{\mathrm{tnr}}}\left(\frac{P_{n,m}^{\mathrm{M}} - P_{n,m}^{\mathrm{T}}(\beta_r,\sigma)}{P_{n,m}^{\mathrm{M}}}\right)^2 \quad \text{where} \quad P^{\mathrm{M}} = \sigma_{\mathrm{N}}\sqrt{\frac{m}{2\pi}}\sum_{k=1}^{N} D^{\mathrm{M}}(t_k).$$

Here, $P_{n,m}^{\mathrm{T}}(\beta_r, \sigma)$ denotes the theoretical price of the swaption under the Hull-White model, and $t_1, \cdots, t_N$ denote the cash flow dates of the underlying swap.

[3] https://www.federalreserve.gov/releases/h15/

3.2 Credit Spread Process, Prepayment Process

In this subsection, we describe the estimation method for the initial values of the credit spread and prepayment processes, $w_0 := w(t_0), , x_0 := x(t_0), , y_0 := y(t_0)$. The parameter set for the latent processes, $\theta := (\alpha_w, \alpha_x, \alpha_y, \beta_w, \beta_x, \beta_y, \sigma_w, \sigma_x, \sigma_y, \rho_{r,x}, \rho_{r,y}, \rho_{x,y})^\top$, is taken from the specifications in Chernov, B. R. Dunn, and Longstaff (2017). Note that in order for the CIR processes (Cox, Ingersoll, and Ross 1985) $w(t)$, $x(t)$, and $y(t)$ to remain nonnegative, the Feller conditions $2\alpha_w > \sigma_w^2$, $2\alpha_x > \sigma_x^2$, and $2\alpha_y > \sigma_y^2$ should be satisfied. The parameters used in this paper satisfy these conditions. TBA forward price data are obtained from Bloomberg. For the analysis, we use Fannie Mae 30-year TBA contracts with maturity $n \in \mathcal{T}_{\text{mat}}^{\text{TBA}} := \{0.3\text{mth}, 1.3\text{mth}, 2.3\text{mth}\}$ and mortgage rate $m \in \mathcal{M}$ $:= \{2.5\%, \cdots, 5.0\%\}$ (18 TBA contracts in total). Since no closed-form solution is known for the TBA theoretical forward price expressed in Equation (27), we compute the theoretical forward price $\text{Fwd}^{(\text{T})}(w_0, x_0, y_0, \theta)$ using Monte Carlo simulation. At each time, the values w_0, x_0, and y_0 are chosen to minimize the following loss function between the observed market forward prices and the theoretical forward prices,

$$\sum_{n \in \mathcal{T}_{\text{mat}}^{\text{TBA}}} \sum_{m \in \mathcal{M}} \left(\text{Fwd}_i^{(\text{M})} - \text{Fwd}_i^{(\text{T})}(w_0, x_0, y_0, \theta) \right)^2 .$$

Latent factor modeling in security pricing generally falls into two categories: arbitrage-free models and equilibrium models (Bruce Tuckman 2022). The interest rate model adopted in this paper, the Hull-White model, is arbitrage-free in the sense that the theoretical discount curve given by (7) is constructed to match the discount curve observed in the market.

On the other hand, the parameters included in the credit spread process (8) and the prepayment processes (14) and (15) are estimated using panel data of TBA forward prices (Chernov, B. R. Dunn, and Longstaff 2017). Therefore, the market forward prices and the theoretical forward prices are not necessarily required to match precisely at each time, and in that sense, the credit spread and prepayment processes can be regarded as equilibrium models.

In this paper, in order to investigate whether mispricing opportunities remain for investors to exploit, we assume that market prices mean-revert to theoretical values, which are regarded as representing fair value. Based on this assumption, we construct a trading strategy using the pricing gap δ_i, defined as the difference between the market price and the theoretical price, as the mispricing signal. From this perspective, the equilibrium model is well-suited for our purpose.

As described above, at each time the parameters w_0, x_0, and y_0 are estimated, and for each time and for each TBA contract we then compute the following price difference δ_i.

$$\delta_i := \text{Fwd}_i^{(\text{M})} - \text{Fwd}_i^{(\text{T})}(w_0, x_0, y_0, \theta) \quad i = 1, \cdots, 18. \tag{29}$$

The price difference is used as a mispricing signal for the trading strategy discussed in Section 5.

4 Estimation Results

This section describes the estimation results of the MBS pricing model, as well as the characteristics and explanatory factors of the price difference defined by Equation (29), which is the difference between the market price and the theoretical price.

We begin with the results in the cross section. Figure 6 shows the market prices and theoretical prices in the cross section. From Figure 6, several observations can be made. First, the relationship between market prices and moneyness differs across time points: on January 31, 2014 and January 31, 2018, market prices exhibit convex and concave shapes with respect to moneyness, respectively, whereas on July 31, 2020, the relationship follows an S-shaped pattern. Second, the degree of deviation of market prices from theoretical prices also varies over time. For example, on January 31, 2018 and July 31, 2020, the deviations are relatively small and large across all levels of moneyness, respectively. In contrast, there is little variation in both the moneyness dependence and the deviation from theoretical prices across different remaining maturities.

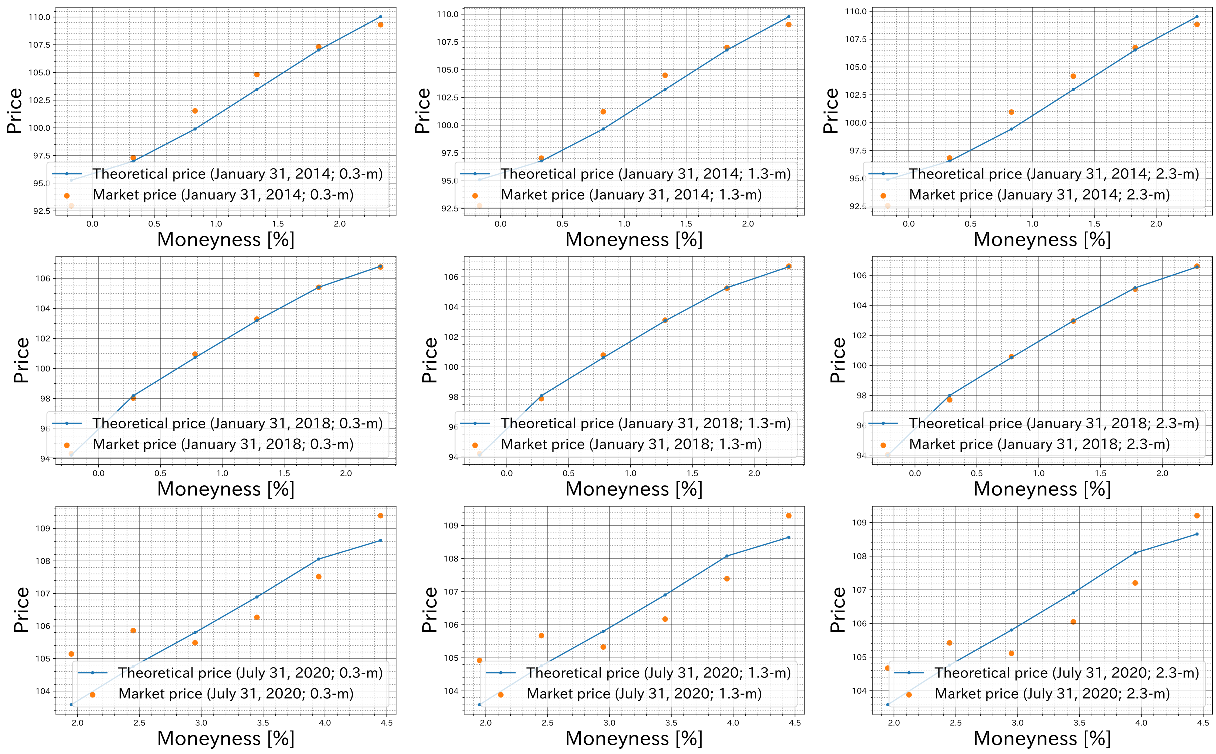

Figure 6: Market prices (orange dots) and theoretical prices (blue dots on the solid lines) in each cross section. The horizontal axis represents moneyness (in percent), and the vertical axis represents the forward price. Moneyness is defined as the mortgage rate of the TBA contract minus the 10-year U.S. Treasury zero yield at each point in time. The figure displays market prices and theoretical prices for TBA forward contracts with approximately 0.3-month, 1.3-month, and 2.3-month settlement horizons as of January 31, 2014; January 31, 2018; and July 31, 2020.

Next, we present the time-series results. Figure 7 shows the time-series of the difference between market prices and theoretical prices, as defined in Equation (29), based on the estimated values of w_0, x_0, and y_0. From Figure 7, it can be observed that the price differentials exhibit mean-reverting behavior across all TBA contracts regardless of mortgage rate or remaining maturity. Additionally, for certain mortgage rates and periods (e.g., 2.5% from 2019 to 2022), a trend component appears in addition to mean reversion. Moreover, price differentials across forward contracts with the same mortgage rate but different remaining maturities generally show little variation, except during specific periods. Assuming that the evolution of the price differential shown in Figure 7 follows an AR(1) process $\delta_{i,t} = \phi_0 + \phi_1 \delta_{i,t-1} + \varepsilon_t$, the half-life, computed as $\ln(0.5)/\ln(|\phi_1|)$, is on average 0.72 months across all TBA contracts. This suggests that investment strategies based on the price differential defined in Equation (29) would require rebalancing at a monthly or even higher frequency.

This section investigates whether the price difference defined in Equation (29) can be explained by certain factors. Figure 8 shows a scatter plot of moneyness and the price difference. Figure 8 shows that the price difference exhibits an inverse smile pattern with respect to moneyness. In other words, TBA contracts that are at-the-money tend to be relatively overpriced compared to their theoretical prices, whereas deep in-the-money and deep out-of-the-money contracts tend to be relatively underpriced. Boyarchenko, Fuster, and Lucca (2019) show that the option-adjusted spread (OAS) exhibits a smile pattern with respect to moneyness. Given the intuition that contracts with higher OAS tend to have a larger discrepancy between theoretical and market OAS, the findings in this paper are consistent with their results. It should be noted, however, that while Boyarchenko, Fuster, and Lucca (2019) use OAS in their analysis, this paper focuses on prices. As a result, a smile pattern appears in their analysis, while an inverse smile pattern is observed in this one. These findings are nevertheless consistent.

To further identify the explanatory factors for the price difference defined in Equation (29), we perform the following regression analysis, where the price difference serves as the dependent variable.

$$\begin{aligned}
\delta_{i,t} &= \beta_0 + \beta_1 \delta_{i,t-1} + \sum_{I \in \mathcal{I}} \mathbb{1}_{\{m_i - R_{10,t} \in I\}} \beta_I^{(\mathrm{mny})} + \sum_{m=2.5\%}^{4.5\%} \mathbb{1}_{\{m_i = m\}} \beta_m^{(\mathrm{mtg})} \\
&\quad + \sum_{T=0.3\mathrm{m}}^{1.3\mathrm{m}} \mathbb{1}_{\{T_i = T\}} \beta_T^{(\mathrm{mat})} + \beta_2 \mathrm{BidAsk}_{i,t} + \epsilon_i. \qquad (30) \\
\text{where} \quad \mathcal{I} &:= \{[-\infty, -1.0\%)], [-1.0\%, 0.0\%), [0.0\%, 1.0\%), [1.0\%, 2.0\%), [2.0\%, 3.0\%)\}.
\end{aligned}$$

Here, $i\,(= 1, \cdots, 18)$ denotes the type of TBA forward contract (for example, a TBA forward contract with approximately 0.3 months to settlement and a mortgage rate of 3.5%). $\delta_{i,t}$ and $\delta_{i,t-1}$ represent the difference between the market and theoretical forward prices of TBA contract i at time t and $t-1$, respectively. m_i is the mortgage rate of contract i (e.g., $m_i = 3.5\%$ for a TBA with a 3.5% coupon), T_i is the remaining months to settlement (e.g., $T_i = 0.3\mathrm{m}$ for a TBA with approximately 0.3 months to settlement), $R_{10,t}$ is the observed 10-year U.S. Treasury zero yield at time t, and $\mathrm{BidAsk}_{i,t}$ is the bid-ask spread of contract i at time t. Explanatory variables related to moneyness, mortgage rate, and time to settlement are treated as dummy variables. The dummy variables on the right-hand side of Equation (30) include moneyness $\mathcal{I}$, mortgage rates $m = 2.5\%, 3.0\%, \cdots, 4.5\%$, and maturities $T = 0.3\mathrm{m}, 1.3\mathrm{m}$. Moneyness is modeled as a categorical (dummy) variable rather than a continuous one in order to capture the nonlinear relationship between moneyness and price differences observed in Figure 8. To account for autocorrelation in $\delta_{i,t}$, we include a lag term $\delta_{i,t-1}$ on the right-hand side of Equation (30). The regression analysis is conducted on panel data covering 18 types of TBA forward contracts (six mortgage rates and three maturities), over the sample period from June 28, 2013 to February 29, 2024. The regression results are presented in Table 2. The results in Table 2 show that the statistically significant explanatory variables include the price difference at time $t-1$, as well as dummy variables for moneyness and mortgage rate. In addition to the moneyness dependence suggested in Figure 8, this implies that the time-series structure of price differences and the mortgage rate also play an important role in explaining the price difference. The coefficients for moneyness, $\beta_I^{(\mathrm{mny})}$ $(I \in \mathcal{I})$, are positive near the at-the-money region and negative further away, which

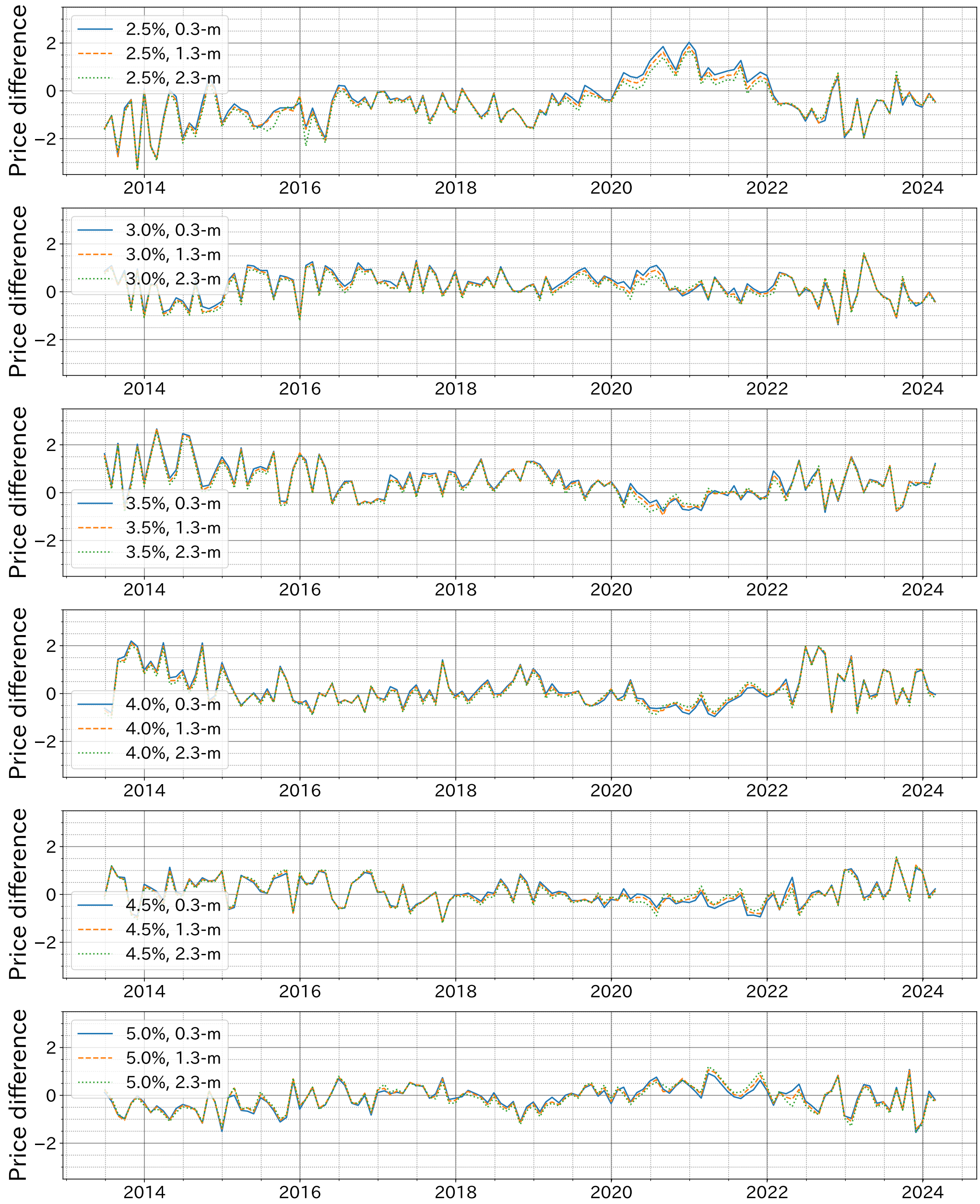

Figure 7: Time-series of the difference between market prices and theoretical prices as defined in Equation (29). From the top to the bottom panels, the figures correspond to TBA contracts with mortgage rates of 2.5%, 3.0%, 3.5%, 4.0%, 4.5%, and 5.0%, respectively. In each panel, the solid blue line, dashed orange line, and dotted green line represent TBA contracts with approximately 0.3-month, 1.3-month, and 2.3-month times to settlement, respectively.

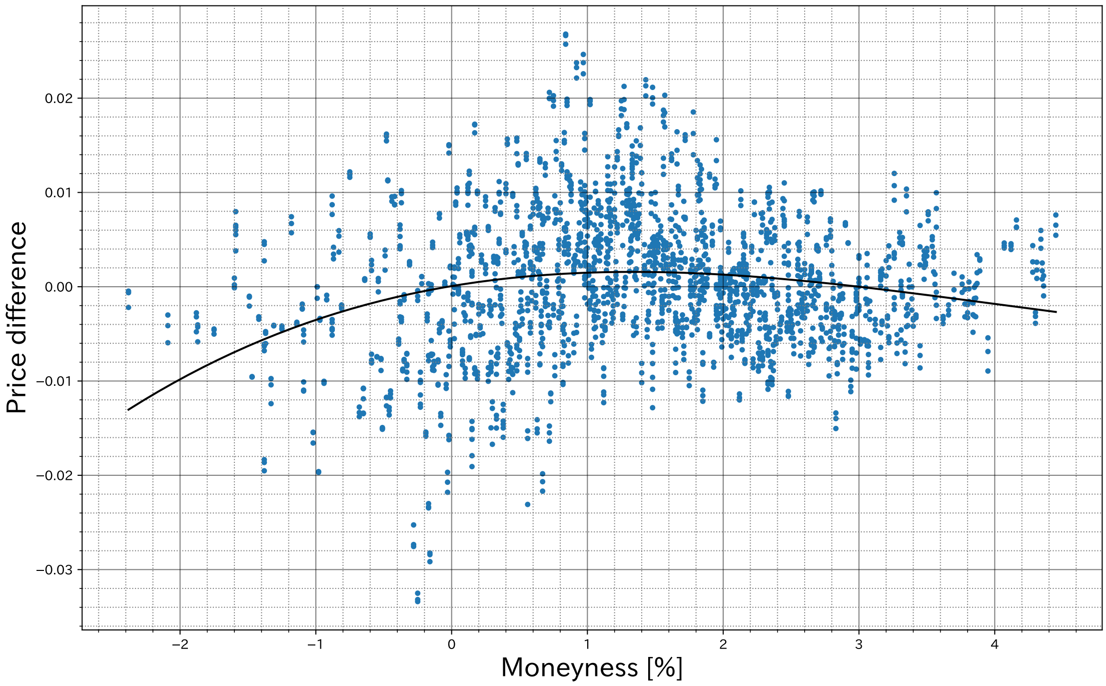

Figure 8: A scatter plot of moneyness (horizontal axis) and the price difference defined in Equation (29) (vertical axis). Moneyness is defined as the difference between the mortgage rate of the TBA contract and the 10-year U.S. Treasury zero yield at each point in time. The solid line represents a cubic polynomial fit to the scatter plot. The data period spans from June 28, 2013 to February 29, 2024.

Table 2: Regression results for the price difference. The regression analysis uses Equation (30). $i, (= 1, \cdots, 18)$ represents the type of TBA forward contract (for example, a TBA forward contract with approximately 0.3 months to settlement and a mortgage rate of $3.5\delta_{i,t}$ and $\delta_{i,t-1}$ denote the difference between the market and theoretical forward prices of contract i at times t and $t-1$, respectively. m_i is the mortgage rate of contract i (e.g., $m_i = 3.5\%$ for a TBA contract with a 3.5% coupon), T_i is the remaining months to settlement (e.g., $T_i = 0.3$m for a contract with approximately 0.3 months remaining), $R_{10,t}$ is the observed 10-year U.S. Treasury zero yield at time t, and $\text{BidAsk}_{i,t}$ is the bid-ask spread of contract i at time t. Asterisks $*$ next to t-statistics indicate significance at the 5% level. The sample period is from June 28, 2013 to February 29, 2024.

Explanatory variable	Coefficient	Coefficient (value)	t-statistic
Intercept	β_0	−0.000712	−1.48
Price difference at time $t-1$	β_1	0.323799	17.17*
Moneyness	$\beta^{(\text{mny})}_{[-\infty,-1.0\%)}$	−0.000555	−0.59
	$\beta^{(\text{mny})}_{[-1.0\%,0.0\%)}$	−0.002331	−3.39*
	$\beta^{(\text{mny})}_{[0.0\%,1.0\%)}$	0.001542	2.64*
	$\beta^{(\text{mny})}_{[1.0\%,2.0\%)}$	0.001692	3.15*
	$\beta^{(\text{mny})}_{[2.0\%,3.0\%)}$	−0.001395	−2.79*
Mortgage rate	$\beta^{(\text{cpn})}_{2.5\%}$	−0.003129	−5.43*
	$\beta^{(\text{mtg})}_{3.0\%}$	0.001179	2.19*
	$\beta^{(\text{mtg})}_{3.5\%}$	0.002844	5.55*
	$\beta^{(\text{mtg})}_{4.0\%}$	0.001108	2.30*
	$\beta^{(\text{mtg})}_{4.5\%}$	0.001033	2.27*
Maturity	$\beta^{(\text{mat})}_{0.3\text{m}}$	0.000513	1.64
	$\beta^{(\text{mat})}_{1.3\text{m}}$	0.000304	0.98
Bid-ask spread	β_2	−0.007460	−0.07
Adjusted R^2			0.320
Sample size			2286

is consistent with the results shown in Figure 8. Meanwhile, remaining maturity and bid-ask spread are not statistically significant explanatory variables for the price difference.

5 Arbitrage Trading Strategy

In this section, we construct an arbitrage trading strategy by using the difference between the market price and the theoretical price defined in Equation (29) as an indicator of mispricing (undervaluation or overvaluation) in investment. Let N denote the total number of tradable TBA contracts. We define the sensitivities of the theoretical price with respect to r_0, w_0, x_0, and y_0 as $D_{r,i}$, $D_{w,i}$, $D_{x,i}$, and $D_{y,i}$, respectively.

$$D_{X,i} := -\frac{1}{\mathrm{Fwd}_i}\frac{\partial \mathrm{Fwd}_i}{\partial X} \quad (X = r_0, w_0, x_0, y_0;\ i = 1, \cdots, N).$$

The benchmark portfolio is defined as an equally weighted portfolio of the tradable TBA contracts. Let $\boldsymbol{D}^{(\mathrm{b})} := (D_r^{(\mathrm{b})}, D_w^{(\mathrm{b})}, D_x^{(\mathrm{b})}, D_y^{(\mathrm{b})})^\top \in \mathbb{R}^{4\times 1}$ denote the sensitivity vector of the benchmark portfolio with respect to r_0, w_0, x_0, and y_0. Here, $D_r^{(\mathrm{b})}$, $D_w^{(\mathrm{b})}$, $D_x^{(\mathrm{b})}$, and $D_y^{(\mathrm{b})}$ represent the sensitivities of the benchmark portfolio to r_0, w_0, x_0, and y_0, respectively, and are calculated as the arithmetic averages of the sensitivities across the N TBA contracts included in the portfolio. The portfolio weights of the arbitrage strategy, denoted by $\boldsymbol{w} \in \mathbb{R}^{N\times 1}$, are determined at each time by solving the following minimization problem.

$$\min_{\boldsymbol{w}} \boldsymbol{\delta}^\top \cdot \boldsymbol{w} \qquad \text{subject to} \qquad \begin{cases} D\boldsymbol{w} = \boldsymbol{D}^{(\mathrm{b})}, \\ |w_i| \le w_{\max} \quad (\forall i = 1, \cdots, N), \\ \sum_{i=1}^N w_i = 1. \end{cases} \tag{31}$$

Here, the vector $\boldsymbol{\delta} \in \mathbb{R}^{N\times 1}$ and the matrix $D \in \mathbb{R}^{4\times N}$ are defined as follows, where each $\delta_i, (i = 1, \cdots, N)$ is defined in Equation (29).

$$\boldsymbol{\delta} := \begin{pmatrix} \delta_1 \\ \delta_2 \\ \vdots \\ \delta_N \end{pmatrix}, \quad D := \begin{pmatrix} D_{r,1} & D_{r,2} & \cdots & D_{r,N} \\ D_{w,1} & D_{w,2} & \cdots & D_{w,N} \\ D_{x,1} & D_{x,2} & \cdots & D_{x,N} \\ D_{y,1} & D_{y,2} & \cdots & D_{y,N} \end{pmatrix}.$$

In Equation (31), the term $\boldsymbol{\delta}^\top \cdot \boldsymbol{w}$ represents the degree of mispricing at the portfolio level, and the portfolio weights are determined so as to minimize this value. When the weights $\boldsymbol{w}$ satisfy the constraint $D\boldsymbol{w} = \boldsymbol{D}^{(\mathrm{b})}$, the portfolio can fully offset the uncertainty arising from the Brownian motions $W_r^{\mathbb{Q}}$, $W_w^{\mathbb{Q}}$, $W_x^{\mathbb{Q}}$, and $W_y^{\mathbb{Q}}$ relative to the benchmark. That is, the portfolio can hedge interest rate risk, credit risk, and prepayment risk in comparison with the benchmark (see Section A.5 for a proof). We set $w\max = 0.5$. Table 3 reports the summary statistics of the excess returns of the arbitrage strategy defined in Equation (31) relative to the equally weighted benchmark portfolio. Table 3 indicates the

Table 3: Summary statistics of the excess return of the arbitrage trading strategy defined in Equation (31) relative to the equally weighted benchmark portfolio. The average, standard deviation, maximum, and minimum values are expressed in annualized basis points (bps). Returns are calculated on a monthly basis, and the data period spans from June 28, 2013, to January 31, 2024. The maximum and minimum values were recorded on December 31, 2014, and October 30, 2020, respectively.

Statistic	Value	Statistic	Value
Mean	58.27	Skewness	1.033
t-value	1.83	Kurtosis	3.236
Standard deviation	103.12	Maximum	1646.50
Information ratio	0.57	Minimum	-726.90

following results. The average excess return is 58.27 bps per year, which is substantially smaller than that of the MBS arbitrage strategy presented in Duarte, Longstaff, and Yu (2006). Since the skewness is positive, the return distribution is fat-tailed in the positive direction, providing evidence against the hypothesis that arbitrage strategies earn small positive returns for most of the time but occasionally suffer large negative returns. Figure 9 displays the time-series evolution of cumulative excess returns relative to the equally weighted benchmark portfolio generated by the arbitrage strategy defined in Equation (31). Figure 9 shows that positive excess returns were generated from 2014 to early 2016

Figure 9: Cumulative excess returns relative to the equally weighted benchmark portfolio constructed by the arbitrage strategy defined in Equation (31). Returns are calculated on a monthly basis over the sample period from June 28, 2013 to January 31, 2024.

and from the end of 2019 to the end of 2020, while little excess return was achieved during other periods. The figure also indicates that interest rate risk appears to be reasonably hedged relative to the benchmark. In fact, during the rising interest rate environment in the latter half of 2021, the excess returns relative to the benchmark did not decline significantly.

6 Conclusion and Discussion

This paper investigates whether persistent mispricing remains in the TBA market, a liquid market for U.S. agency MBSs, using a no-arbitrage pricing framework. A mean-reverting time-series pattern is observed in the difference between the market forward prices of 30-year Fannie Mae TBAs and the model-implied forward prices under the no-arbitrage model. Under the no-arbitrage pricing model used in this paper, the price difference exhibits an inverse smile pattern with respect to moneyness: TBAs that are at-the-money tend to be overvalued, while those deep in- or out-of-the-money tend to be undervalued. Boyarchenko, Fuster, and Lucca (2019) describe that the option-adjusted spread (OAS) exhibits a smile pattern with respect to moneyness. Given the intuition that TBAs with larger OAS tend to show larger deviations between theoretical and market OAS, the inverse smile pattern in price differences found in this paper is consistent with their findings. The regression analysis also reveals that in addition to moneyness, the time-series dynamics of the pricing difference and the mortgage rate play important roles in explaining the mispricing. The arbitrage trading strategy that uses the pricing difference as a mispricing indicator earns an average annualized excess return of 58.27 basis points relative to an equally weighted benchmark portfolio. However, the time-series of excess returns reveals that there are periods when the strategy earns positive excess returns and other periods when it earns almost none. The following discusses these results and outlines directions for future research. One possible reason for the alternation between high and low return periods is that the one-factor interest rate model used in the analysis may not fully hedge interest rate risk. To address this issue, it would be desirable to extend the current one-factor model to a multi-factor interest rate model. In this study, a simple arbitrage strategy is constructed based solely on the cross-sectional price difference, under the assumption that mispricing reverts toward zero without remaining underpriced or overpriced for extended periods. However, the empirical results indicate that pricing differences can persist or follow trends beyond typical mean-reversion timescales. Therefore, it may be worthwhile to incorporate historical averages, standard deviations, and half-lives of the pricing difference to adjust the mean-reversion center or to better time position entry and exit. Furthermore, it remains unclear whether the observed excess returns are indeed driven by convergence of the pricing differences. To identify the sources of excess returns, it would be useful to decompose the returns into contributions attributable to interest rate risk, credit risk, and prepayment risk.

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References

Ang, Andrew and Francis A. Longstaff (2013). "Systemic sovereign credit risk: Lessons from the U.S. and Europe". In: *Journal of Monetary Economics* 60.5. Aggregate Implications of Local Public Finance, pp. 493–510. ISSN: 0304-3932. DOI: https://doi.org/10.1016/j.jmoneco.2013.04.009.

Boyarchenko, Nina, Andreas Fuster, and David O Lucca (Feb. 2019). "Understanding Mortgage Spreads". In: *The Review of Financial Studies* 32.10, pp. 3799–3850. ISSN: 0893-9454. DOI: 10.1093/rfs/hhz004.

Brennan, Michael J. and Eduardo S. Schwartz (1985). "Determinants of GNMA Mortgage Prices". In: *Real Estate Economics* 13.3, pp. 209–228. DOI: https://doi.org/10.1111/1540-6229.00351.

Brigo, D. and F. Mercurio (2013). *Interest Rate Models Theory and Practice*. Springer Finance. Springer Berlin Heidelberg. ISBN: 9783662045534. URL: https://books.google.co.jp/books?id=USvrCAAAQBAJ.

Bruce Tuckman, Angel Serrat (2022). *Fixed Income Securities: Tools for Today's Markets*. Springer Finance. John Wiley & Sons, Ltd. ISBN: 9781119835622.

Capone, C.A. (2001). "Introduction to the Special Issue on Mortgage Modeling". In: *The Journal of Real Estate Finance and Economics* 23, pp. 131–137. ISSN: 10527001. DOI: https://doi.org/10.1023/A:1011162231116. URL: http://www.jstor.org/stable/24832827 (visited on 11/18/2024).

Carlin, Bruce I., Francis A. Longstaff, and Kyle Matoba (2014). "Disagreement and asset prices". In: *Journal of Financial Economics* 114.2, pp. 226–238. ISSN: 0304-405X. DOI: https://doi.org/10.1016/j.jfineco.2014.06.007.

Chaudhary, S (Oct. 2006). "The trust IO/PO market. RMBS Trading Strategy". In: *Bank of America*, pp. 1–29.

Chernov, Mikhail, Brett R. Dunn, and Francis A. Longstaff (Dec. 2017). "Macroeconomic-Driven Prepayment Risk and the Valuation of Mortgage-Backed Securities". In: *The Review of Financial Studies* 31.3, pp. 1132–1183. ISSN: 0893-9454. DOI: https://doi.org/10.1093/rfs/hhx140.

Cheyette, Oren (1996). "Implied Prepayments". In: *The Journal of Portfolio Management* 23.2, pp. 107–115. DOI: 10.3905/jpm.1996.409575.

Cox, John C., Jonathan E. Ingersoll, and Stephen A. Ross (1985). "A Theory of the Term Structure of Interest Rates". In: *Econometrica* 53.2, pp. 385–407. ISSN: 00129682, 14680262. DOI: https://doi.org/10.2307/1911242. (Visited on 11/15/2024).

Deng, Yongheng, John M. Quigley, and Robert Van Order (2000). "Mortgage Terminations, Heterogeneity and the Exercise of Mortgage Options". In: *Econometrica* 68.2, pp. 275–307. DOI: https://doi.org/10.1111/1468-0262.00110.

Diep, Peter, Andrea L. Eisfeldt, and Scott Richardson (2021). "The Cross Section of MBS Returns". In: *The Journal of Finance* 76.5, pp. 2093–2151. DOI: https://doi.org/10.1111/jofi.13055.

Duarte, Jefferson, Francis A. Longstaff, and Fan Yu (July 2006). "Risk and Return in Fixed-Income Arbitrage: Nickels in Front of a Steamroller?" In: *The Review of Financial Studies* 20.3, pp. 769–811. ISSN: 0893-9454. DOI: 10.1093/rfs/hhl026.

Duffie, Darrell and Kenneth J. Singleton (1997). "An Econometric Model of the Term Structure of Interest-Rate Swap Yields". In: *The Journal of Finance* 52.4, pp. 1287–1321. DOI: https://doi.org/10.1111/j.1540-6261.1997.tb01111.x.

— (June 1999). "Modeling Term Structures of Defaultable Bonds". In: *The Review of Financial Studies* 12.4, pp. 687–720. ISSN: 0893-9454. DOI: 10.1093/rfs/12.4.687.

Dunn, Kenneth B. and John J. McConnell (1981a). "A Comparison of Alternative Models for Pricing GNMA Mortgage-Backed Securities". In: *The Journal of Finance* 36.2, pp. 471–484. DOI: https://doi.org/10.1111/j.1540-6261.1981.tb00463.x.

— (1981b). "Valuation of GNMA Mortgage-Backed Securities". In: *The Journal of Finance* 36.3, pp. 599–616. DOI: https://doi.org/10.1111/j.1540-6261.1981.tb00647.x.

Dunn, Kenneth B. and Chester S. Spatt (2005). "The Effect of Refinancing Costs and Market Imperfections on the Optimal Call Strategy and the Pricing of Debt Contracts". In: *Real Estate Economics* 33.4, pp. 595–617. DOI: https://doi.org/10.1111/j.1540-6229.2005.00132.x.

Fabozzi, Frank J. (2016). *The Handbook of Mortgage-Backed Securities: 7th Edition*. Oxford University Press. ISBN: 9780198785774.

Frank J. Fabozzi, Vinod Kothari (2008). "Structuring Agency MBS Deals". In: *Introduction to Securitization*. John Wiley & Sons, Ltd. Chap. 3, pp. 29–64. ISBN: 9781118266892. DOI: https://doi.org/10.1002/9781118266892.ch3.

Gabaix, Xavier, Arvind Krishnamurthy, and Olivier Vigneron (2007). "Limits of Arbitrage: Theory and Evidence from the Mortgage-Backed Securities Market". In: *The Journal of Finance* 62.2, pp. 557–595. DOI: https://doi.org/10.1111/j.1540-6261.2007.01217.x.

Hayre, Lakhbir (2001). *Salomon Smith Barney Guide to Mortgage-Backed and Asset-Backed Securities*. John Wiley. ISBN: 9780471385875.

Hull, John and Alan White (Oct. 1990). "Pricing Interest-Rate-Derivative Securities". In: *The Review of Financial Studies* 3.4, pp. 573–592. ISSN: 0893-9454. DOI: 10.1093/rfs/3.4.573.

Kau, James B. and Donald C. Keenan (1995). "An Overview of the Option-Theoretic Pricing of Mortgages". In: *Journal of Housing Research* 6.2, pp. 217–244. ISSN: 10527001. URL: http://www.jstor.org/stable/24832827 (visited on 11/18/2024).

Kijima, M. and I. Nagayama (Apr. 1994). "Efficient numerical procedures for the Hull-White extended Vasicek model". In: *Journal of Financial Engineering* 3.2, pp. 275–292.

Levin, Alexander and Andrew Davidson (2005). "Prepayment Risk-and Option-Adjusted Valuation of MBS". In: *The Journal of Portfolio Management* 31.4, pp. 73–85. DOI: 10.3905/jpm.2005.570153.

Longstaff, Francis A., Sanjay Mithal, and Eric Neis (2005). "Corporate Yield Spreads: Default Risk or Liquidity? New Evidence from the Credit Default Swap Market". In: *The Journal of Finance* 60.5, pp. 2213–2253. DOI: https://doi.org/10.1111/j.1540-6261.2005.00797.x.

Longstaff, Francis A., Jun Pan, et al. (Apr. 2011). "How Sovereign Is Sovereign Credit Risk?" In: *American Economic Journal: Macroeconomics* 3.2, pp. 75–103. DOI: 10.1257/mac.3.2.75.

Pan, Jun and Kenneth J. Singleton (2008). "Default and Recovery Implicit in the Term Structure of Sovereign CDS Spreads". In: *The Journal of Finance* 63.5, pp. 2345–2384. DOI: https://doi.org/10.1111/j.1540-6261.2008.01399.x.

Richard, Scott F. and Richard Roll (1989). "Prepayments on fixed-rate mortgage-backed securities". In: *The Journal of Portfolio Management* 15.3, pp. 73–82. DOI: 10.3905/jpm.1989.409207.

Schwartz, Eduardo S. and Walter N. Torous (1989). "Prepayment and the Valuation of Mortgage-Backed Securities". In: *The Journal of Finance* 44.2, pp. 375–392. DOI: https://doi.org/10.1111/j.1540-6261.1989.tb05062.x.

— (1992). "Prepayment, Default, and the Valuation of Mortgage Pass-through Securities". In: *The Journal of Business* 65.2, pp. 221–39. URL: https://EconPapers.repec.org/RePEc:ucp:jnlbus:v:65:y:1992:i:2:p:221-39.

— (1993). "Mortgage Prepayment and Default Decisions: A Poisson Regression Approach". In: *Real Estate Economics* 21.4, pp. 431–449. DOI: https://doi.org/10.1111/1540-6229.00619.

Stanton, Richard and Nancy Wallace (1998). "Mortgage Choice: What's the Point?" In: *Real Estate Economics* 26.2, pp. 173–205. DOI: https://doi.org/10.1111/1540-6229.00743.

Turan Bali, Massoud Heidari and Liuren Wu (2009). "Predictability of Interest Rates and Interest-Rate Portfolios". In: *Journal of Business & Economic Statistics* 27.4, pp. 517–527. DOI: 10.1198/jbes.2009.06124.

Vickery, James Ian and Joshua Wright (2013). "TBA Trading and Liquidity in the Agency MBS Market". In: *Federal Reserve Bank of New York Economic Policy Review* 19.1, pp. 1–18. DOI: http://dx.doi.org/10.2139/ssrn.2324222.

Wallace, Nancy (2005). "Innovations in Mortgage Modeling: An Introduction". In: *Real Estate Economics* 33, pp. 587–593. URL: https://api.semanticscholar.org/CorpusID:54613969.

A Proofs and Remarks

A.1 Derivation of Equations(9)–(11)

We define $P_w(t,T)$ as follows.

$$P_w(t,T) := \mathbb{E}_t^{\mathbb{Q}}\left[e^{-\int_t^T w(s)\,\mathrm{d}s}\right] \tag{32}$$

Since the survival probability $S(t)$ is given by $S(t) = P_w(0,t)$, it suffices to solve for $P_w(t,T)$. From Itô's formula and the no-arbitrage condition, $P_w(t,T)$ satisfies the following partial differential equation.

$$\frac{\partial P_w}{\partial t}(t,T) + (\alpha_w - \beta_w w)\frac{\partial P_w}{\partial w}(t,T) + \frac{1}{2}\sigma_w^2 w \frac{\partial^2 P_w}{\partial w^2}(t,T) = wP_w(t,T) \tag{33}$$

$$P_w(T,T) = 1 \tag{34}$$

The boundary condition in Equation (34) is obtained from Equation (32). Letting $\tau := T - t$, we consider the following ansatz for the solution of $P_w(t,T)$

$$P_w(t,T) := e^{A_w(\tau) - B_w(\tau)w(t)} \tag{35}$$

Substituting this into Equations (33) and (34) and simplifying, we obtain the following equations.

$$\left(B_w'(\tau) + \beta_w B_w(\tau) + \frac{1}{2}\sigma_w^2 B_w^2(\tau) - 1\right)w - (A_w'(\tau) + \alpha_w B_w(\tau)) = 0 \tag{36}$$

$$e^{A_w(0) - B_w(0)w(T)} = 1 \tag{37}$$

Here, the prime symbol $'$ denotes differentiation with respect to τ. For Equations (36)–(37) to hold for any $w \in \mathbb{R}$, the following ordinary differential equations must be satisfied.

$$A_w'(\tau) + \alpha_w B_w(\tau) = 0 \tag{38}$$

$$B_w'(\tau) + \beta_w B_w(\tau) + \frac{1}{2}\sigma_w^2 B_w^2(\tau) - 1 = 0 \tag{39}$$

$$A_w(0) = B_w(0) = 0 \tag{40}$$

Solving Equations (38)–(40) gives the following.

$$A_w(\tau) = \frac{2\alpha_w}{\sigma_w^2} \ln \left[\frac{2h\, e^{(\beta_w+h)\tau/2}}{2h + (\beta_w + h)(e^{h\tau} - 1)} \right] \tag{41}$$

$$B_w(\tau) = \frac{2(e^{h\tau} - 1)}{2h + (\beta_w + h)(e^{h\tau} - 1)} \tag{42}$$

Substituting $t = 0$ and $\tau = t$ into Equations (35) and (41)–(42) yields Equations (9)–(11).

A.2 Derivation of Equations(17)–(26)

In this section, following Hayre (2001) and Chernov, B. R. Dunn, and Longstaff (2017), we derive the discrete-time cash flow expressions under monthly compounding, namely Equations (17)–(26), from the continuous-time cash flow expression under continuous compounding given in Equation (5). Along with the discretization of cash flows, we switch from continuous compounding to monthly compounding (12 times per year). Let $t_0\,(=0)$ denote the current time, and let the cash flow dates be represented by $\{t_1, t_2, \cdots, t_K\}$, where each interval $t_i - t_{i-1}$ $(i = 1, \cdots, K)$ is one month (i.e., $\frac{1}{12}$ year). As in the continuous case, we first consider the case without prepayment. The scheduled monthly payment at time t_i $(i = 1, \cdots, K)$, denoted by $\mathrm{PAY}(t_i)$, is obtained by the following discretization.

$$c\,\mathrm{d}t = \frac{m}{1 - e^{-mT}}\mathrm{d}t \overset{\text{discretization}}{\rightarrow} \mathrm{PAY}(t_i) = \frac{\frac{m}{12}}{1 - \left(1 + \frac{m}{12}\right)^{-K}} \quad \text{where} \quad K := mT. \tag{43}$$

Next, the remaining balance at time t_i $(i = 1, \cdots, K)$, denoted by $\mathrm{BAL}(t_i)$, is obtained by the following discretization using Equation (2).

$$\tilde{P}(t_i) = \frac{1 - e^{-m(T-t_i)}}{1 - e^{-mT}} \overset{\text{discretization}}{\rightarrow} \mathrm{BAL}(t_i) = \frac{1 - \left(1 + \frac{m}{12}\right)^{-K+i}}{1 - \left(1 + \frac{m}{12}\right)^{-K}}. \tag{44}$$

Furthermore, the principal portion of the payment at time t_i $(i = 1, \cdots, K)$, denoted by $\mathrm{PRIN}(t_i)$, is given by the difference in remaining balance between t_{i-1} and t_i, and is therefore obtained as follows.

$$-\mathrm{d}\tilde{P}(t) \overset{\text{discretization}}{\rightarrow} \mathrm{PRIN}(t_i) = \mathrm{BAL}(t_{i-1}) - \mathrm{BAL}(t_i) = \frac{\frac{m}{12} \cdot \left(1 + \frac{m}{12}\right)^{-K+i-1}}{1 - \left(1 + \frac{m}{12}\right)^{-K}} = \mathrm{PAY}(t_i) \cdot \left(1 + \frac{m}{12}\right)^{-K+i-1}. \tag{45}$$

The interest portion of the payment, denoted by $\mathrm{INT}(t_i)$, is given by the following expression.

$$\mathrm{INT}(t_i) = \mathrm{PAY}(t_i) - \mathrm{PRIN}(t_i). \tag{46}$$

Next, we consider the case with prepayment. Let $\widetilde{\mathrm{PRIN}}(t_i)$, $\widetilde{\mathrm{INT}}(t_i)$, and $\mathrm{PP}(t_i)$ denote the scheduled principal payment, scheduled interest payment, and principal prepaid at time t_i $(i = 1, \cdots, K)$, respectively. Then, the cash flow $\mathrm{CF}(t_i)$ at time t_i is given by the following expression.

$$\mathrm{CF}(t_i) = \widetilde{\mathrm{PRIN}}(t_i) + \widetilde{\mathrm{INT}}(t_i) + \mathrm{PP}(t_i). \tag{47}$$

In actual MBS transactions, investors receive as cash flows the principal and interest payments made by mortgage borrowers, excluding servicing fees and guaranty fees. Let g denote the servicing and guaranty fee rate. Then, the MBS cash flows are given by the following modified version of Equation (47), which reflects the exclusion of these fees.

$$\mathrm{CF}(t_i) = \widetilde{\mathrm{PRIN}}(t_i) + \frac{m-g}{m}\widetilde{\mathrm{INT}}(t_i) + \mathrm{PP}(t_i). \tag{48}$$

Let τ_i $(i = 1, \cdots, K)$ denote the time at which the cash flow amount at time t_i is determined. For 30-year MBSs issued by FNMA, τ_i is approximately one month prior to t_i, as shown in Table 1. Figure 10 illustrates the relationship between t_i and τ_i, as well as the remaining principal factor used to calculate the cash flow amount. Here, the remaining principal factor is defined by $N(t) := e^{-\int_0^t \lambda(u)\,\mathrm{d}u}$. The calculation of the cash flow amount at time t_i uses the remaining principal factor over the period $[\tau_{i-1}, \tau_i)$. For the scheduled principal and interest payments at time t_i—that is, the sum of the scheduled principal payment (dark blue area in the figure) and the scheduled interest payment (light blue area)—the remaining principal factor at τ_{i-1}, $N(\tau_{i-1})$, is used. For the

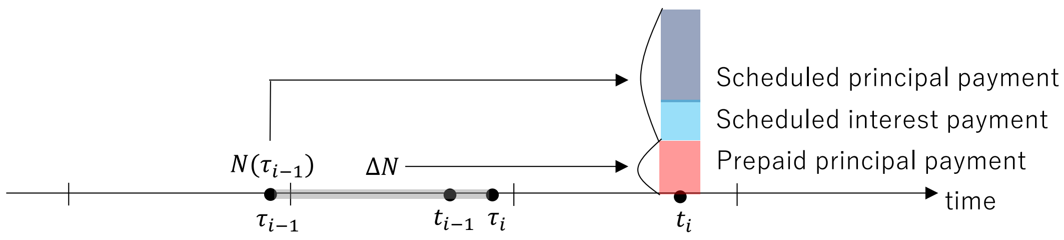

Figure 10: The figure shows the relationship between t_i and τ_i, and the remaining principal factor used to calculate the cash flow amount. The calculation of the cash flow at time t_i uses the remaining principal factor over the period $[\tau_{i-1}, \tau_i)$. For the scheduled principal and interest payments at time t_i, that is, the sum of the scheduled principal payment (dark blue area in the figure) and the scheduled interest payment (light blue area), the remaining principal factor $N(\tau_{i-1})$ is used. For the prepaid principal payment at time t_i (red area in the figure), the change in the principal factor over the period $[\tau_{i-1}, \tau_i)$, denoted by $\Delta N = N(\tau_i) - N(\tau_{i-1})$, is used. Therefore, the time at which the cash flow amount paid at t_i is determined is τ_i.

prepaid principal payment at time t_i (red area in the figure), the change in the principal factor over the period $[\tau_{i-1}, \tau_i)$, denoted by $\Delta N = N(\tau_i) - N(\tau_{i-1})$, is used. Therefore, the cash flow amount paid at t_i is determined at time τ_i. The scheduled principal payment and scheduled interest payment at each time t_i ($i = 1, \ldots, K$) are given by the following expressions.

$$\widetilde{\mathrm{PRIN}}(t_i) = N(\tau_{i-1})\,\mathrm{PRIN}(t_i), \tag{49}$$

$$\widetilde{\mathrm{INT}}(t_i) = N(\tau_{i-1})\,\mathrm{INT}(t_i). \tag{50}$$

The remaining balance $\widetilde{\mathrm{BAL}}(t_i)$ at time t_i in the case where prepayments occur is defined as follows.

$$\widetilde{\mathrm{BAL}}(t_i) := N(\tau_i)\,\mathrm{BAL}(t_i). \tag{51}$$

$\widetilde{\mathrm{BAL}}(t_i)$ is defined as the remaining balance at time t_i in the presence of prepayment. It should be emphasized that $\widetilde{\mathrm{BAL}}(t_i)$ reflects prepayments that occurred before time τ_i. $\widetilde{\mathrm{PRIN}}(t_i)$, $\widetilde{\mathrm{INT}}(t_i)$, and $\widetilde{\mathrm{BAL}}(t_i)$ use the remaining principal factor evaluated at different times. The former two are based on $N(\tau_{i-1})$, while the latter is based on $N(\tau_i)$. The single monthly mortality (SMM) rate is defined by the following equation.

$$\mathrm{SMM}(\tau_i) := \frac{N(\tau_{i-1}) - N(\tau_i)}{N(\tau_{i-1})}. \tag{52}$$

The prepaid principal payment at time t_i for $i = 1, \cdots, K$ is obtained by the following discretization.

$$\begin{aligned}
P(t) \cdot \lambda(t)\mathrm{d}t &= \tilde{P}(t) \cdot e^{-\int_0^t \lambda(u)\mathrm{d}u} \lambda(t)\mathrm{d}t \\
&= \tilde{P}(t) \cdot (-\mathrm{d}N(t)) \\
&\overset{\text{discretization}}{\rightarrow} \mathrm{PP}(t_i) \\
&= \mathrm{BAL}(t_i) \cdot [N(\tau_{i-1}) - N(\tau_i)] \\
&= [\mathrm{BAL}(t_{i-1}) - \mathrm{PRIN}(t_i)] \cdot [N(\tau_{i-1}) - N(\tau_i)] \\
&= N(\tau_{i-1}) \cdot [\mathrm{BAL}(t_{i-1}) - \mathrm{PRIN}(t_i)] \cdot \frac{N(\tau_{i-1}) - N(\tau_i)}{N(\tau_{i-1})} \\
&= [N(\tau_{i-1})\mathrm{BAL}(t_{i-1}) - N(\tau_{i-1})\mathrm{PRIN}(t_i)] \cdot \mathrm{SMM}(\tau_i) \\
&= \left[\widetilde{\mathrm{BAL}}(t_{i-1}) - \widetilde{\mathrm{PRIN}}(t_i)\right] \mathrm{SMM}(\tau_i).
\end{aligned}$$

The transformation from the first to the second line uses Itô' s formula applied to the remaining principal factor, yielding $\mathrm{d}N(t) = -N(t)\lambda(t)\mathrm{d}t$. The transformation from the fourth to the fifth line uses Equation (45), from the sixth to the seventh line uses the definition of the single monthly mortality rate given in Equation (52), and from the seventh to the eighth line uses Equations (49) and (51). Based on these derivations, Equations (17)–(26) are obtained.

A.3 Remark on Equation(12)

In Section 2.2, we assume that the recovery rate is given by $\tilde{\delta}(t) \equiv 0$ and use the corresponding MBS pricing formula given in Equation (12). In this subsection, we discuss the expression for MBS pricing under a more general assumption for the recovery rate. Under the mathematical framework described in Section 2.2, let $\tilde{\delta}(t)$ be an $\mathcal{F}_t$-predictable process representing the recovery rate applied to the MBS price just prior to default at time t. Then, the theoretical MBS price process $\left\{V^{\mathrm{MBS}}(t)\right\}$ is given by

$$\begin{aligned} V^{\mathrm{MBS}}(t) &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r(u)\,\mathrm{d}u}\mathbb{1}_{\{\tau>s\}}\,\mathrm{dCF}(s) + \mathbb{1}_{\{t<\tau\le T\}}e^{-\int_t^\tau r(u)\,\mathrm{d}u}\tilde{\delta}(\tau)V^{\mathrm{MBS}}(\tau-)\middle|\,\mathcal{G}_t\right] \\ &= \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s r(u)\,\mathrm{d}u}(1-\mathbb{1}_{\{\tau\le s\}})\,\mathrm{dCF}(s) + \int_t^T e^{-\int_t^s r(u)\,\mathrm{d}u}\tilde{\delta}(s)V^{\mathrm{MBS}}(s-)\,\mathrm{d}\mathbb{1}_{\{\tau\le s\}}\middle|\,\mathcal{G}_t\right] \end{aligned} \tag{53}$$

Let $\tilde{V}(t) := V^{\mathrm{MBS}}(t-)$. It follows that $V^{\mathrm{MBS}}(t)\mathbb{1}_{\{\tau>t\}} = \tilde{V}(t)\mathbb{1}_{\{\tau>t\}}$, and $\left\{\tilde{V}(t)\right\}$ can be regarded as an $\mathcal{F}_t$-predictable process. We define the stochastic process $\{G(t)\}$ as follows

$$\begin{aligned} G(t) = e^{-\int_0^t r(u)\,\mathrm{d}u}V^{\mathrm{MBS}}(t)\left(1-\mathbb{1}_{\{\tau\le t\}}\right) &+ \int_0^t e^{-\int_0^s r(u)\,\mathrm{d}u}\left(1-\mathbb{1}_{\{\tau\le s\}}\right)\mathrm{dCF}(s) \\ &+ \int_0^t e^{-\int_0^s r(u)\,\mathrm{d}u}\tilde{\delta}(s)\tilde{V}(s)\,\mathrm{d}\mathbb{1}_{\{\tau\le s\}} \end{aligned} \tag{54}$$

The process $\{G(t)\}$ can be interpreted as the discounted value process of the MBS and it can be shown to be a $(\mathbb{Q},\{\mathcal{G}_t\})$-martingale. Noting that the process $\mathbb{1}_{\{\tau\le t\}} - \int_0^t\left(1-\mathbb{1}_{\{\tau\le s\}}\right)w(s)\,\mathrm{d}s$ is a $(\mathbb{Q},\{\mathcal{G}_t\})$-martingale, we apply Itô's formula to Equation (54) and obtain

$$\begin{aligned} \mathrm{d}G(t) &= -r(t)e^{-\int_0^t r(u)\,\mathrm{d}u}V^{\mathrm{MBS}}(t)(1-\mathbb{1}_{\{\tau\le t\}})\,\mathrm{d}t + e^{-\int_0^t r(u)\,\mathrm{d}u}(1-\mathbb{1}_{\{\tau\le t\}})\,\mathrm{d}V^{\mathrm{MBS}}(t) \\ &\quad - e^{-\int_0^t r(u)\,\mathrm{d}u}V^{\mathrm{MBS}}(t-)\,\mathrm{d}\mathbb{1}_{\{\tau\le t\}} \\ &\quad + e^{-\int_0^t r(u)\,\mathrm{d}u}(1-\mathbb{1}_{\{\tau\le t\}})\,\mathrm{dCF}(t) + e^{-\int_0^t r(u)\,\mathrm{d}u}\tilde{\delta}(t)\tilde{V}(t)\,\mathrm{d}\mathbb{1}_{\{\tau\le t\}} \\ &= e^{-\int_0^t r(u)\,\mathrm{d}u}(1-\mathbb{1}_{\{\tau\le t\}})\,\mathrm{d}\tilde{V}(t) - e^{-\int_0^t r(u)\,\mathrm{d}u}\left[r(t)+\left(1-\tilde{\delta}(t)\right)w(t)\right]\tilde{V}(t)(1-\mathbb{1}_{\{\tau\le t\}})\,\mathrm{d}t \\ &\quad + e^{-\int_0^t r(u)\,\mathrm{d}u}(1-\mathbb{1}_{\{\tau\le t\}})\,\mathrm{dCF}(t) + e^{-\int_0^t r(u)\,\mathrm{d}u}\tilde{\delta}(t)\tilde{V}(t)\left[\mathrm{d}\mathbb{1}_{\{\tau\le t\}} - (1-\mathbb{1}_{\{\tau\le t\}})w(t)\,\mathrm{d}t\right] \end{aligned}$$

Since the variation of $\tilde{V}(t)$ can be considered on the set $\{\tau>t\}$, we have

$$\begin{aligned} \mathrm{d}\tilde{V}(t) = &\left[r(t)+\left(1-\tilde{\delta}(t)\right)w(t)\right]\tilde{V}(t)\,\mathrm{d}t - \mathrm{dCF}(t) \\ &+ e^{\int_0^t r(u)\,\mathrm{d}u}\,\mathrm{d}G(t) - \tilde{\delta}(t)\tilde{V}(t)\left[\mathrm{d}\mathbb{1}_{\{\tau\le t\}} - (1-\mathbb{1}_{\{\tau\le t\}})w(t)\,\mathrm{d}t\right] \end{aligned} \tag{55}$$

The sum of the last two terms on the right-hand side of Equation (55) can be regarded as the differential of a $(\mathbb{Q},\{\mathcal{G}_t\})$-martingale $\{m(t)\}$, defined as

$$\mathrm{d}m(t) := e^{\int_0^t r(u)\,\mathrm{d}u}\,\mathrm{d}G(t) - \tilde{\delta}(t)\tilde{V}(t)\left[\mathrm{d}\mathbb{1}_{\{\tau\le t\}} - (1-\mathbb{1}_{\{\tau\le t\}})w(t)\,\mathrm{d}t\right]$$

Substituting this into Equation (55), we obtain

$$\mathrm{d}\tilde{V}(t) = \left[r(t)+\left(1-\tilde{\delta}(t)\right)w(t)\right]\tilde{V}(t)\,\mathrm{d}t - \mathrm{dCF}(t) + \mathrm{d}m(t) \tag{56}$$

Equation (56) can be regarded as a linear stochastic differential equation in $\tilde{V}(t)$, and we solve it accordingly. Specifically, from Equation (56), we observe that

$$\mathrm{d}\left(e^{-\int_0^t\left[r(u)+\left(1-\tilde{\delta}(u)\right)w(u)\right]\mathrm{d}u}\tilde{V}(t)\right) = e^{-\int_0^t\left[r(u)+\left(1-\tilde{\delta}(u)\right)w(u)\right]\mathrm{d}u}\left(-\mathrm{dCF}(t)+\mathrm{d}m(t)\right)$$

From this, we obtain the following expression

$$\begin{aligned} &\tilde{V}(T) = e^{\int_t^T\left[r(u)+\left(1-\tilde{\delta}(u)\right)w(u)\right]\mathrm{d}u}\tilde{V}(t) + \int_t^T e^{-\int_T^s\left[r(u)+\left(1-\tilde{\delta}(u)\right)w(u)\right]\mathrm{d}u}\left(-\mathrm{dCF}(s)+\mathrm{d}m(s)\right) \\ \iff\; &e^{-\int_t^T\left[r(u)+\left(1-\tilde{\delta}(u)\right)w(u)\right]\mathrm{d}u}\tilde{V}(T) = \tilde{V}(t) + \int_t^T e^{-\int_t^s\left[r(u)+\left(1-\tilde{\delta}(u)\right)w(u)\right]\mathrm{d}u}\left(-\mathrm{dCF}(s)+\mathrm{d}m(s)\right) \end{aligned}$$

Noting that $\tilde{V}(T) = 0$, we obtain the following

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_t^T \left[r(u)+\left(1-\tilde{\delta}(u)\right)w(u)\right] \mathrm{d}u} \tilde{V}(T) \middle| \mathcal{F}_t \right] = 0, \\
\iff \quad & \mathbb{E}^{\mathbb{Q}}\left[\tilde{V}(t) + \int_t^T e^{-\int_t^s \left[r(u)+\left(1-\tilde{\delta}(u)\right)w(u)\right] \mathrm{d}u} \left(-\mathrm{dCF}(s) + \mathrm{d}m(s)\right) \middle| \mathcal{F}_t \right] = 0, \\
\iff \quad & \tilde{V}(t) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s \left[r(u)+\left(1-\tilde{\delta}(u)\right)w(u)\right] \mathrm{d}u} \mathrm{dCF}(s) \middle| \mathcal{F}_t \right] \\
& \qquad - \mathbb{E}^{\mathbb{Q}}\left[\mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s \left[r(u)+\left(1-\tilde{\delta}(u)\right)w(u)\right] \mathrm{d}u} \mathrm{d}m(s) \middle| \mathcal{G}_t \right] \middle| \mathcal{F}_t \right] \qquad (57)
\end{aligned}
$$

Since the second term on the right-hand side of Equation (57) is a stochastic integral with respect to a $(\mathbb{Q}, \{\mathcal{G}_t\})$-martingale, it vanishes in expectation and we obtain

$$
\tilde{V}(t) = \mathbb{E}^{\mathbb{Q}}\left[\int_t^T e^{-\int_t^s \left[r(u)+\left(1-\tilde{\delta}(u)\right)w(u)\right] \mathrm{d}u} \mathrm{dCF}(s) \middle| \mathcal{F}_t \right]
$$

In the main analysis, we assume that the recovery rate is $\tilde{\delta}(t) \equiv 0$, so Equation (53) reduces to Equation (12).

A.4 Remark on Equation(27)

In this subsection, we derive the theoretical forward price under continuous time when the underlying asset is the actual MBS price, denoted by $V^{\mathrm{MBS}}(t)$. The forward price $\mathrm{Fwd}(t_0, t_s)$ is defined as an $\mathcal{F}_{t_0}$-measurable random variable that satisfies the following condition.

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{t_s} r(u)\mathrm{d}u} \left(\mathrm{Fwd}(t_0, t_s) - V^{\mathrm{MBS}}(t_s)\right) \middle| \mathcal{F}_{t_0} \right] = 0, \\
\Leftrightarrow\, & \mathrm{Fwd}(t_0, t_s) = \frac{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{t_s} r(u)\mathrm{d}u} V^{\mathrm{MBS}}(t_s) \middle| \mathcal{F}_{t_0} \right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{t_s} r(u)\mathrm{d}u} \middle| \mathcal{F}_{t_0} \right]}. \qquad (58)
\end{aligned}
$$

Here, noting that

$$
e^{-\int_0^t r(u)\,\mathrm{d}u} \left(V^{\mathrm{MBS}}(t) + \int_0^t e^{\int_s^t r(u)\,\mathrm{d}u} \mathrm{dCF}(s) \right)
$$

is a $\mathbb{Q}$-martingale, the numerator of the right-hand side of Equation (58) can be rewritten as follows.

$$
\begin{aligned}
& \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{t_s} r(u)\mathrm{d}u} V^{\mathrm{MBS}}(t_s) \middle| \mathcal{F}_{t_0} \right] \\
&= \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{t_s} r(u)\mathrm{d}u} \left(V^{\mathrm{MBS}}(t_s) + \int_0^{t_s} e^{\int_s^{t_s} r(u)\mathrm{d}u} \mathrm{dCF}(s) \right) \middle| \mathcal{F}_{t_0} \right] - \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{t_s} r(u)\mathrm{d}u} \int_0^{t_s} e^{\int_s^{t_s} r(u)\mathrm{d}u} \mathrm{dCF}(s) \middle| \mathcal{F}_{t_0} \right] \\
&= e^{\int_0^{t_0} r(u)\mathrm{d}u} \mathbb{E}^{\mathbb{Q}}\left[e^{-\int_0^{t_s} r(u)\mathrm{d}u} \left(V^{\mathrm{MBS}}(t_s) + \int_0^{t_s} e^{\int_s^{t_s} r(u)\mathrm{d}u} \mathrm{dCF}(s) \right) \middle| \mathcal{F}_{t_0} \right] - \mathbb{E}^{\mathbb{Q}}\left[\int_0^{t_s} e^{-\int_{t_0}^{s} r(u)\mathrm{d}u} \mathrm{dCF}(s) \middle| \mathcal{F}_{t_0} \right] \\
&= e^{\int_0^{t_0} r(u)\mathrm{d}u} \left[e^{-\int_0^{t_0} r(u)\mathrm{d}u} \left(V^{\mathrm{MBS}}(t_0) + \int_0^{t_0} e^{\int_s^{t_0} r(u)\mathrm{d}u} \mathrm{dCF}(s) \right) \right] - \mathbb{E}^{\mathbb{Q}}\left[\int_0^{t_s} e^{-\int_{t_0}^{s} r(u)\mathrm{d}u} \mathrm{dCF}(s) \middle| \mathcal{F}_{t_0} \right] \\
&= V^{\mathrm{MBS}}(t_0) + \int_0^{t_0} e^{-\int_{t_0}^{s} r(u)\mathrm{d}u} \mathrm{dCF}(s) - \int_0^{t_0} e^{-\int_{t_0}^{s} r(u)\mathrm{d}u} \mathrm{dCF}(s) - \mathbb{E}^{\mathbb{Q}}\left[\int_0^{t_s} e^{-\int_{t_0}^{s} r(u)\mathrm{d}u} \mathrm{dCF}(s) \middle| \mathcal{F}_{t_0} \right] \\
&= V^{\mathrm{MBS}}(t_0) - \mathbb{E}^{\mathbb{Q}}\left[\int_0^{t_s} e^{-\int_{t_0}^{s} r(u)\mathrm{d}u} \mathrm{dCF}(s) \middle| \mathcal{F}_{t_0} \right].
\end{aligned}
$$

Thus we have the following expression.

$$
\mathrm{Fwd}(t_0, t_s) = \frac{V^{\mathrm{MBS}}(t_0)}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{t_s} r(u)\mathrm{d}u} \middle| \mathcal{F}_{t_0} \right]} - \frac{\mathbb{E}^{\mathbb{Q}}\left[\int_0^{t_s} e^{-\int_{t_0}^{s} r(u)\mathrm{d}u} \mathrm{dCF}(s) \middle| \mathcal{F}_{t_0} \right]}{\mathbb{E}^{\mathbb{Q}}\left[e^{-\int_{t_0}^{t_s} r(u)\mathrm{d}u} \middle| \mathcal{F}_{t_0} \right]} \qquad (59)
$$

The following considerations lead to the derivation of Equation (27). In the TBA market, it is important to account for the fact that the quoted forward price reflects the price for delivering \$100 of MBS principal at the settlement date t_s. At time t_s, the remaining principal balance of the underlying asset $V^{\mathrm{MBS}}(t)$ is given by $e^{-\int_{t_0}^{t_s}\lambda(u)\,\mathrm{d}u}\cdot \mathrm{BAL}(t_s)$. Therefore, one should consider a forward contract on a principal-adjusted underlying asset, specifically $\dfrac{100\cdot e^{\int_{t_0}^{t_s}\lambda(u)\,\mathrm{d}u}\cdot V^{\mathrm{MBS}}(t)}{\mathrm{BAL}(t_s)}$.

A.5 Conditions on Portfolio Weights for Risk Hedging

Here we derive that, for a portfolio composed of TBA forward contracts, the condition that the portfolio weights must satisfy in order to fully hedge the portfolio' s value changes—relative to the benchmark—against the uncertainty arising from the risk factors $W_r^{\mathbb{Q}}$, $W_w^{\mathbb{Q}}$, $W_x^{\mathbb{Q}}$, and $W_y^{\mathbb{Q}}$ is given by the constraint $D\boldsymbol{w} = \boldsymbol{D}^{(\mathrm{b})}$ in Equation (31). In the following derivation, carry terms proportional to $\mathrm{d}t$ are neglected. Let $\mathrm{Fwd}_i(t)$ denote the forward price of TBA contract $i\,(i=1,\ldots,N)$ at time t. Let $\mathrm{dFwd}_i(t)$ represent the price change over the infinitesimal interval $(t, t+\mathrm{d}t]$, and define the return as $\mathrm{d}R_i(t) := \frac{\mathrm{dFwd}_i(t)}{\mathrm{Fwd}_i(t)}$. We assume that no cash flows from the underlying assets occur during the interval $(t, t+\mathrm{d}t]$. Using Itô's lemma, the forward price change $\mathrm{dFwd}_i(t)$ is approximately expressed as follows.

$$\mathrm{dFwd}_i(t) \simeq \frac{\partial \mathrm{Fwd}_i}{\partial r}\mathrm{d}r(t) + \frac{\partial \mathrm{Fwd}_i}{\partial w}\mathrm{d}w(t) + \frac{\partial \mathrm{Fwd}_i}{\partial x}\mathrm{d}x(t) + \frac{\partial \mathrm{Fwd}_i}{\partial y}\mathrm{d}y(t).$$

Here, we ignore carry terms proportional to $\mathrm{d}t$, and simplify expressions such as $\frac{\partial \mathrm{Fwd}_i}{\partial \xi}(t,\xi,w(t),x(t),y(t))\Big|_{\xi=r(t)}$ by denoting them as $\frac{\partial \mathrm{Fwd}_i}{\partial r}$, and so on. We define the sensitivities of the forward price with respect to $r(t)$, $w(t)$, $x(t)$, and $y(t)$ as follows

$$D_{r,i}(t) := -\frac{1}{\mathrm{Fwd}_i}\frac{\partial \mathrm{Fwd}_i}{\partial r}, \tag{60}$$

$$D_{w,i}(t) := -\frac{1}{\mathrm{Fwd}_i}\frac{\partial \mathrm{Fwd}_i}{\partial w}, \tag{61}$$

$$D_{x,i}(t) := -\frac{1}{\mathrm{Fwd}_i}\frac{\partial \mathrm{Fwd}_i}{\partial x}, \tag{62}$$

$$D_{y,i}(t) := -\frac{1}{\mathrm{Fwd}_i}\frac{\partial \mathrm{Fwd}_i}{\partial y}. \tag{63}$$

Then, the return $\mathrm{d}R_i(t)$ of the TBA contract i can be expressed as follows.

$$\begin{aligned}
\mathrm{d}R_i(t) &\simeq -\left(-\frac{1}{\mathrm{Fwd}_i}\frac{\partial \mathrm{Fwd}_i}{\partial r}\right)\mathrm{d}r(t) - \left(-\frac{1}{\mathrm{Fwd}_i}\frac{\partial \mathrm{Fwd}_i}{\partial w}\right)\mathrm{d}w(t) \\
&\quad - \left(-\frac{1}{\mathrm{Fwd}_i}\frac{\partial \mathrm{Fwd}_i}{\partial x}\right)\mathrm{d}x(t) - \left(-\frac{1}{\mathrm{Fwd}_i}\frac{\partial \mathrm{Fwd}_i}{\partial y}\right)\mathrm{d}y(t) \\
&= -D_{r,i}(t)\mathrm{d}r(t) - D_{w,i}(t)\mathrm{d}w(t) - D_{x,i}(t)\mathrm{d}x(t) - D_{y,i}(t)\mathrm{d}y(t) \\
&\simeq -D_{r,i}(t)\,\sigma_r \mathrm{d}W_r^{\mathbb{Q}}(t) - D_{w,i}(t)\sigma_w \mathrm{d}W_w^{\mathbb{Q}}(t) - D_{x,i}(t)\sigma_x\sqrt{x(t)}\mathrm{d}W_x^{\mathbb{Q}}(t) - D_{y,i}(t)\sigma_y\sqrt{y(t)}\mathrm{d}W_y^{\mathbb{Q}}(t).
\end{aligned}$$

Here, from the third to the fourth line, we used Equations (6), (8), (14), and (15). Accordingly, the vector of forward price returns $\mathrm{d}\boldsymbol{R}(t) := (\mathrm{d}R_1(t_0),\cdots,\mathrm{d}R_N(t_0))^\top \in \mathbb{R}^{N\times 1}$ can be expressed as

$$\begin{aligned}
\mathrm{d}\boldsymbol{R}(t) &\simeq \begin{pmatrix}
-D_{r,1}(t)\sigma_r & -D_{w,1}(t)\sigma_w & -D_{x,1}(t)\sigma_x\sqrt{x(t)} & -D_{y,1}(t)\sigma_y\sqrt{y(t)} \\
-D_{r,2}(t)\sigma_r & -D_{w,2}(t)\sigma_w & -D_{x,2}(t)\sigma_x\sqrt{x(t)} & -D_{y,2}(t)\sigma_y\sqrt{y(t)} \\
\vdots & \ddots & \ddots & \vdots \\
-D_{r,N}(t)\sigma_r & -D_{w,N}(t)\sigma_w & -D_{x,N}(t)\sigma_x\sqrt{x(t)} & -D_{y,N}(t)\sigma_y\sqrt{y(t)}
\end{pmatrix}
\begin{pmatrix}\mathrm{d}W_r^{\mathbb{Q}}(t)\\ \mathrm{d}W_w^{\mathbb{Q}}(t)\\ \mathrm{d}W_x^{\mathbb{Q}}(t)\\ \mathrm{d}W_y^{\mathbb{Q}}(t)\end{pmatrix} \\
&= -\begin{pmatrix}
D_{r,1}(t) & D_{w,1}(t) & D_{x,1}(t) & D_{y,1}(t) \\
D_{r,2}(t) & D_{w,2}(t) & D_{x,2}(t) & D_{y,2}(t) \\
\vdots & \ddots & \ddots & \vdots \\
D_{r,N}(t) & D_{w,N}(t) & D_{x,N}(t) & D_{y,N}(t)
\end{pmatrix}
\begin{pmatrix}
\sigma_r & 0 & 0 & 0 \\
0 & \sigma_w & 0 & 0 \\
0 & 0 & \sigma_x\sqrt{x(t)} & 0 \\
0 & 0 & 0 & \sigma_y\sqrt{y(t)}
\end{pmatrix}
\begin{pmatrix}\mathrm{d}W_r^{\mathbb{Q}}(t)\\ \mathrm{d}W_w^{\mathbb{Q}}(t)\\ \mathrm{d}W_x^{\mathbb{Q}}(t)\\ \mathrm{d}W_y^{\mathbb{Q}}(t)\end{pmatrix}.
\end{aligned}$$

and we can find the following expression.

$$\mathrm{d}\boldsymbol{R}(t) \simeq D^\top \Sigma\,\mathrm{d}\boldsymbol{W}^{\mathbb{Q}}(t). \tag{64}$$

Here, $\boldsymbol{W}^{\mathbb{Q}}(t) \in \mathbb{R}^{4\times 1}$ and $D \in \mathbb{R}^{4\times N}$, $\Sigma \in \mathbb{R}^{4\times 4}$ are defined as follows.

$$\boldsymbol{W}^{\mathbb{Q}}(t) := \begin{pmatrix} W_r^{\mathbb{Q}}(t) \\ W_w^{\mathbb{Q}}(t) \\ W_x^{\mathbb{Q}}(t) \\ W_y^{\mathbb{Q}}(t) \end{pmatrix}, \quad D := \begin{pmatrix} D_{r,1}(t) & D_{r,2}(t) & \cdots & D_{r,N}(t) \\ D_{w,1}(t) & D_{w,2}(t) & \cdots & D_{w,N}(t) \\ D_{x,1}(t) & D_{x,2}(t) & \cdots & D_{x,N}(t) \\ D_{y,1}(t) & D_{y,2}(t) & \cdots & D_{y,N}(t) \end{pmatrix}, \quad \Sigma := \begin{pmatrix} \sigma_r & 0 & 0 & 0 \\ 0 & \sigma_w & 0 & 0 \\ 0 & 0 & \sigma_x\sqrt{x(t)} & 0 \\ 0 & 0 & 0 & \sigma_y\sqrt{y(t)} \end{pmatrix}.$$

Let $\boldsymbol{w} := (w_1(t), \cdots, w_N(t))^\top \in \mathbb{R}^{N\times 1}$ denote the portfolio weights constructed from TBA forward contracts. Then, the rate of change in the portfolio value over a small time interval $(t, t+\mathrm{d}t]$ can be expressed as follows using Equation (64) and the fact that $\Sigma^\top = \Sigma$.

$$\boldsymbol{w}^\top \cdot \mathrm{d}\boldsymbol{R}(t) \simeq -\left[\Sigma D \boldsymbol{w}\right]^\top \cdot \mathrm{d}\boldsymbol{W}^{\mathbb{Q}}(t). \tag{65}$$

From the above, it follows that the condition that the portfolio weights must satisfy in order to fully hedge against changes in portfolio value due to the uncertainty of the risk factors, namely the Brownian motions $W_r^{\mathbb{Q}}$, $W_w^{\mathbb{Q}}$, $W_x^{\mathbb{Q}}$, and $W_y^{\mathbb{Q}}$, is given by the following expression from Equation (65).

$$\Sigma D \boldsymbol{w} = \boldsymbol{0}.$$

Since all the diagonal elements of Σ are greater than zero, the inverse matrix Σ^{-1} =diag$(\sigma_r^{-1}, \sigma_w^{-1}, (\sigma_x\sqrt{x(t)})^{-1}, (\sigma_y\sqrt{y(t)})^{-1})$ exists, and we obtain the following.

$$D\boldsymbol{w} = \boldsymbol{0}. \tag{66}$$

Equation (66) can be interpreted as a generalization of a neutral position with respect to interest rate risk using modified duration in a government bond portfolio. Furthermore, let $\boldsymbol{w}^{(\mathrm{b})}$ denote the benchmark portfolio weight vector. When constructing a duration-neutral position relative to the benchmark, the condition that the portfolio weights must satisfy is given by replacing $\boldsymbol{w}$ in Equation (66) with $\boldsymbol{w} - \boldsymbol{w}^{(\mathrm{b})}$ as follows.

$$D(\boldsymbol{w} - \boldsymbol{w}^{(\mathrm{b})}) = 0 \quad \Leftrightarrow \quad D\boldsymbol{w} = D\boldsymbol{w}^{(\mathrm{b})}.$$